

THE n -BODY PROBLEM IN SPACES OF CONSTANT CURVATURE

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ABSTRACT. We generalize the Newtonian n -body problem to spaces of curvature $\kappa = \text{constant}$, and study the motion in the 2-dimensional case. For $\kappa > 0$, the equations of motion encounter non-collision singularities, which occur when two bodies are antipodal. This phenomenon leads, on one hand, to hybrid solution singularities for as few as 3 bodies, whose corresponding orbits end up in a collision-antipodal configuration in finite time; on the other hand, it produces non-singularity collisions, characterized by finite velocities and forces at the collision instant. We also point out the existence of several classes of relative equilibria, including the hyperbolic rotations for $\kappa < 0$. In the end, we prove Saari's conjecture when the bodies are on a geodesic that rotates elliptically or hyperbolically. We also emphasize that fixed points are specific to the case $\kappa > 0$, hyperbolic relative equilibria to $\kappa < 0$, and Lagrangian orbits of arbitrary masses to $\kappa = 0$ —results that provide new criteria towards understanding the large-scale geometry of the physical space.

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1. INTRODUCTION

The goal of this paper is to extend the Newtonian n -body problem of celestial mechanics to spaces of constant curvature. Though attempts of this kind existed for two bodies in the 19th century, they faded away after the birth of special and general relativity, to be resurrected several decades later, but only in the case $n = 2$. As we will further argue, the topic we are opening here is important for understanding particle dynamics in spaces other than Euclidean, for shedding some new light on the classical case, and perhaps helping us understand the nature of the physical space.

1.1. History of the problem. The first researcher who took the idea of gravitation beyond \mathbf{R}^3 was Nikolai Lobachevsky. In 1835, he proposed a Kepler problem in the 3-dimensional hyperbolic space, \mathbf{H}^3 , by defining an attractive force proportional to the inverse area of the 2-dimensional sphere of the same radius as the distance between bodies, [52]. Independently of him, and at about the same time, János Bolyai came up with a similar idea, [4].

These co-discoverers of the first non-Euclidean geometry had no followers in their pre-relativistic attempts until 1860, when Paul Joseph Serret¹ extended the gravitational force to the sphere \mathbf{S}^2 and solved the corresponding Kepler problem, [65]. Ten years later, Ernst Schering revisited Lobachevsky's law for which he obtained an analytic expression given by the cotangent potential we study in this paper, [62]. Schering also wrote that Lejeune Dirichlet had told some friends to have dealt with the same problem during his last years in Berlin², [63]. In 1873, Rudolph Lipschitz considered the same problem in \mathbf{S}^3 , but defined a potential proportional to $1/\sin(r/R)$, where r denotes the distance between bodies and R is the curvature radius, [51]. He obtained the general solution of this problem in terms of elliptic functions, but his failure to provide an explicit formula invited new approaches.

In 1885, Wilhelm Killing adapted Lobachevsky's idea to \mathbf{S}^3 and defined an extension of the Newtonian force given by the inverse area of a 2-dimensional sphere (in the spirit of Schering), for which he proved a generalization of Kepler's three laws, [41]. Another contributor was Heinrich Liebmann,³. In 1902, he showed that the orbits of the two-body problem are conics in \mathbf{S}^3 and \mathbf{H}^3 and generalized Kepler's three laws to $\kappa \neq 0$, [47]. One year later, Liebmann proved \mathbf{S}^2 - and \mathbf{H}^2 -analogues of Bertrand's theorem, [3], [76], which states that there exist only two analytic central potentials in the Euclidean

¹Paul Joseph Serret (1827-1898) should not be confused with another French mathematician, Joseph Alfred Serret (1819-1885), known for the Frenet-Serret formulas of vector calculus.

²This must have happened around 1852, as claimed by Rudolph Lipschitz, [50].

³Although he signed his works as Heinrich Liebmann, his full name was Karl Otto Heinrich Liebmann (1874-1939). He did most of his work in Heidelberg and Munich.

space for which all bounded orbits are closed, [48]. He also summed up his results in a book published in 1905, [49].

Unfortunately, this direction of research was neglected in the decades following the birth of special and general relativity. Starting with 1940, however, Erwin Schrödinger developed a quantum-mechanical analogue of the Kepler problem in \mathbf{S}^2 , [64]. Schrödinger used the same cotangent potential of Schering and Liebmann, which he deemed to be the natural extension of Newton's law to the sphere⁴. Further results in this direction were obtained by Leopold Infeld, [36], [71]. In 1945, Infeld and his student Alfred Schild extended this problem to spaces of constant negative curvature using a potential given by the hyperbolic cotangent of the distance. A list of the above-mentioned works also appears in [66], except for Serret's book, [65]. A bibliography of works about mechanical problems in spaces of constant curvature is given in [69].

Several members of the Russian school of celestial mechanics, including Valeri Kozlov and Alexander Harin, [43], [45], Alexey Borisov, Ivan Mamaev, and Alexander Kilin, [5], [6], [7], [8], [39], Alexey Shchepetilov, [67], [68], [69], and Tatiana Vozmischeva, [74], revisited the idea of the cotangent potential for the 2-body problem and considered related problems in spaces of constant curvature starting with the 1990s. The main reason for which Kozlov and Harin supported this approach was mathematical. They pointed out, as Schering, Liebmann, Schrödinger, Infeld, and others had insisted earlier, that (i) the classical one-body problem satisfies Laplace's equation (i.e. the potential is a harmonic function), which also means that the equations of the problem are equivalent with those of the harmonic oscillator; (ii) its potential generates a central field in which all bounded orbits are closed—according to Bertrand's theorem. Then they showed that the cotangent potential is the only one that satisfies these properties in spaces of constant curvature and has, at the same time, meaning in celestial mechanics. The results they obtained support the idea that the cotangent potential is, so far, the best extension found for the Newtonian potential to spaces of nonzero constant curvature. Our paper brings new arguments that support this view.

The latest contribution to the case $n = 2$ belongs to José Cariñena, Manuel Rañada, and Mariano Santander, who provided a unified approach in the framework of differential geometry, emphasizing the dynamics of the cotangent potential in \mathbf{S}^2 and \mathbf{H}^2 , [9] (see also [10], [33]). They also proved that, in this unified context, the conic orbits known in Euclidean space extend naturally to spaces of constant curvature, in agreement with the results obtained by Liebmann, [66].

⁴“The correct form of [the] potential (corresponding to $1/r$ of the flat space) is known to be $\cot \chi$,” [64], p. 14.

1.2. Relativistic n -body problems. Before trying to approach this problem with contemporary tools, we were compelled to ask why the direction of research proposed by Lobachevsky was neglected after the birth of relativity. Perhaps this phenomenon occurred because relativity hoped not only to answer the questions this research direction had asked, but also to regard them from a better perspective than classical mechanics, whose days seemed to be numbered. Things, however, didn't turn out this way. Research on the classical Newtonian n -body problem continued and even flourished in the decades to come, and the work on the case $n = 2$ in spaces of constant curvature was revived after several decades. But how did relativity fare with respect to this fundamental problem of any gravitational theory?

Although the most important success of relativity was in cosmology and related fields, there were attempts to discretize Einstein's equations and define an n -body problem. Remarkable in this direction were the contributions of Jean Chazy, [13], Tullio Levi-Civita, [44], [46], Arthur Eddington, [27], Albert Einstein, Leopold Infeld⁵, and Banesh Hoffmann, [28], and Vladimir Fock, [30]. Subsequent efforts led to refined post-Newtonian approximations (see, e.g., [15], [16], [17]), which prove useful in practice, from understanding the motion of artificial satellites—a field with applications in geodesy and geophysics—to using the Global Positioning System (GPS), [18].

But the equations of the n -body problem derived from relativity prove complicated even for $n = 2$, and they are not prone to analytical studies similar to the ones done in the classical case. This is probably the reason why the need of some simpler equations revived the research on the motion of two bodies in spaces of constant curvature.

Nobody, however, considered the general n -body problem⁶ for $n \geq 3$. The lack of developments in this direction may again rest with the complicated form the equations of motion take if one starts from the idea of defining the potential in terms of the intrinsic distance in the framework of differential geometry. Such complications might have discouraged all the attempts to generalize the problem to more than two bodies.

1.3. Our approach. The present paper overcomes the above-mentioned difficulties encountered in defining a meaningful n -body problem prone to the same mathematical depth achieved in the classical case, by replacing the differential-geometric approach used for $n = 2$ in the case of the cotangent potential with the variational method of constrained Lagrangian dynamics. Also, the technical complications that arise in understanding the motion within the standard

⁵A vivid description of the collaboration between Einstein and Infeld appears in [37].

⁶One of us (Erensto Pérez-Chavela), together with his student Luis Franco-Pérez, recently analyzed a restricted 3-body problem in \mathbf{S}^1 , [31], in a more restrained context than the one we provide here.

models of the Bolyai-Lobachevsky plane (the Klein-Beltrami disk, the Poincaré upper-half-plane, and the Poincaré disk) are bypassed through the less known Weierstrass hyperboloidal model (see Appendix), which often provides analogies with the results we obtain in the spherical case. This model also allows us to use hyperbolic rotations—a class of isometries—to put into the evidence some unexpected solutions of the equations of motion.

The history of the problem shows that there is no unique way of extending the classical idea of gravitation to spaces of constant curvature, but that the cotangent potential is the most natural candidate. Therefore we take this potential as a starting point of our approach, though some of our results—as for example Saari’s conjecture in the geodesic case—do not use this potential explicitly, only its property of being a homogenous function of degree zero.

Our generalization recovers the Newtonian law when the curvature is zero. Moreover, it provides a unified context, in which the potential varies continuously with the curvature κ . The same continuity occurs for the basic results when the curvature tends to zero. For instance, the set of closed orbits of the Kepler problem on non-zero-curvature surfaces tends to the set of ellipses in the Euclidean plane when $\kappa \rightarrow 0$ (see, e.g., [9] or [47]).

2. SUMMARY OF RESULTS

2.1. Equations of motion. In Section 3, we extend the Newtonian potential of the n -body problem to spaces of constant curvature, κ , for any finite dimension. For $\kappa \neq 0$, the potential turns out to be a homogeneous function of degree zero. We also show the existence of an energy integral as well as of the integrals of the angular momentum. Like in general relativity, there are no integrals of the center of mass and linear momentum. But unlike in relativity, where—in the passage from continuous matter to discrete bodies—the fact that forces don’t cancel at the center of mass leads to difficulties in defining infinitesimal sizes for finite masses, [44], we do not encounter such problems here. We assume that the laws of classical mechanics hold for point masses moving on manifolds, so we can apply the results of constrained Lagrangian dynamics to derive the equations of motion. Thus two kinds of forces act on bodies: (i) those given by the mutual interaction between particles, represented by the gradient of the potential, and (ii) those that occur due to the constraints, which involve both position and velocity terms.

2.2. Singularities. In Section 4 we focus on singularities, and distinguish between singularities of the equations of motion and solution singularities. For any $\kappa \neq 0$, the equations of motion become singular at collisions, the same as in the Euclidean case. The case $\kappa > 0$, however, introduces some new singularities, which we call antipodal because they occur when two bodies are at the opposite ends of a diameter of the sphere.

The set of singularities is endowed with a natural dynamical structure. When the motion of three bodies takes place along a geodesic, solutions close to binary collisions and away from antipodal singularities end up in collision, so binary collisions are attractive. But antipodal singularities are repulsive in the sense that no matter how close two bodies are to an antipodal singularity, they never reach it if the third body is far from a collision with any of them.

Solution singularities arise naturally from the question of existence and uniqueness of initial value problems. For nonsingular initial conditions, standard results of the theory of differential equations ensure local existence and uniqueness of an analytic solution defined in some interval $[0, t^+)$. This solution can be analytically extended to an interval $[0, t^*)$, with $0 < t^+ \leq t^* \leq \infty$. If $t^* = \infty$, the solution is globally defined. If $t^* < \infty$, the solution is called singular and is said to have a singularity at time t^* .

While the existence of solutions ending in collisions is obvious for any value of κ , the occurrence of other singularities is not easy to demonstrate. Nevertheless, we prove that some hybrid singular solutions exist in the 3-body problem with $\kappa > 0$. These orbits end up in finite time in a collision-antipodal singularity. Whether other types of non-collision singularities exist, like the pseudocollisions of the Euclidean case, remains an open question. The main reason why this problem is not easy to answer rests with the nonexistence of the center-of-mass integrals.

Another class of solutions connected to collision-antipodal configurations is particularly interesting. We show that, for $n = 3$, there are orbits that reach such a configuration at some instant t^* but remain analytic at this point because the forces and the velocities involved remain finite at t^* . Such a motion can, of course, be analytically continued beyond t^* . This is the first example of a natural non-singularity collision.

2.3. Relative equilibria. The rest of this paper, except for the Appendix, focuses on the results we obtained in \mathbf{S}^2 and \mathbf{H}^2 , mainly because these two surfaces are representative for the cases $\kappa > 0$ and $\kappa < 0$, respectively. Indeed, the results we proved for these surfaces can be extended to different curvatures of the same sign by a mere change of factor.

Sections 5 and 6 deal with relative equilibria in \mathbf{S}^2 and \mathbf{H}^2 . In \mathbf{S}^2 we only have elliptic relative equilibria. Instead, the relative equilibria in \mathbf{H}^2 are of two kinds: elliptic relative equilibria, generated by elliptic rotations, and hyperbolic relative equilibria, generated by hyperbolic rotations (see Appendix). Parabolic relative equilibria, generated by parabolic rotations, do not exist.

Some of the results we obtain in \mathbf{S}^2 have analogues in \mathbf{H}^2 ; others are specific to each case. Theorems 6 and 10, for instance, are dual to each other, whereas Theorem 2 takes place only in \mathbf{S}^2 . The latter identifies a class of fixed points of the equations of motion. More precisely, we prove that if an odd number

n of equal masses are placed, initially at rest, at the vertices of a regular n -gon inscribed in a great circle, then the bodies won't move. The same is true for four equal masses placed at the vertices of a regular tetrahedron inscribed in \mathbf{S}^2 , but—due to the occurrence of antipodal singularities—fails to hold for the other regular polyhedra: octahedron (6 bodies), cube (8 bodies), dodecahedron (12 bodies), and icosahedron (20 bodies), as well as in the case of geodesic n -gons with an even number of bodies.

Theorem 3 shows that there are no fixed points for n bodies within any hemisphere of \mathbf{S}^2 . Its hyperbolic analogue, stated in Theorem 9, proves the nonexistence of fixed points in \mathbf{H}^2 . These two results are in agreement with the Euclidean case in the sense that the n -body problem has no fixed points within distances, say, not larger than the ray of the visible universe.

It is also natural to ask whether fixed points can generate relative equilibria. Theorem 4 shows that if n masses m_1, m_2, \dots, m_n lie initially on a great circle of \mathbf{S}^2 such that the mutual forces are in equilibrium, then any uniform rotation applied to the system generates a relative equilibrium.

Theorem 5 states that the only way to generate an elliptic relative equilibrium from an initial n -gon configuration taken on a great circle, as in Theorem 2, is to assign suitable velocities in the plane of the n -gon. So a regular polygon of this kind can rotate only in a plane orthogonal to the rotation axis.

Theorem 6 and its hyperbolic analogue, Theorem 10, show that n -gons of any admissible size can rotate on the same circle, both in \mathbf{S}^2 and \mathbf{H}^2 . Again, these results agree with the Euclidean case. But something interesting happens with the equilateral (Lagrangian) solutions. Unlike in Euclidean space, elliptic relative equilibria moving in the same plane of \mathbf{R}^3 can be generated only when the masses move on the same circle and are therefore equal, as we prove in Theorems 7 and 11. Thus Lagrangian solutions with unequal masses are specific to the Euclidean case.

Theorems 8 and 12 show that analogues to the collinear (Eulerian) orbits in the 3-body problem of the classical case exist in \mathbf{S}^2 and \mathbf{H}^2 , respectively. While nothing surprising happens in \mathbf{H}^2 , where we prove the existence of such solutions of any size, an interesting phenomenon takes place in \mathbf{S}^2 . Assume that one body lies on the rotation axis (which contains one height of the triangle), while the other two are at the opposite ends of a rotating diameter on some non-geodesic circle of \mathbf{S}^2 . Then elliptic relative equilibria exist while the bodies are at initial positions within the same hemisphere. When the rotating bodies are placed on the equator, however, they encounter an antipodal singularity. Below the equator, solutions exist again until the bodies form an equilateral triangle. By Theorem 5, any n -gon with an odd number of sides can rotate only in its own plane, so the (vertical) equilateral triangle is a fixed point but cannot lead to an elliptic relative equilibrium. If the rotating bodies are then placed below the equilateral position, solutions fail to exist. But the

masses don't have to be all equal. Eulerian solutions exist if, say, the non-rotating body has mass m and the other two have mass M . If $M \geq 4m$, these orbits occur for all $z \neq 0$. Again, these results prove that, as long as we do not exceed reasonable distances, such as the ray of the visible universe, the behavior of elliptic relative equilibria lying on a rotating geodesic is similar to the one of Eulerian solutions in the Euclidean case.

We then study hyperbolic relative equilibria around a point and along a (usually non-geodesic) hyperbola. Theorem 13 proves that such orbits do not exist on fixed geodesics of \mathbf{H}^2 , so the bodies cannot chase each other along a geodesic while maintaining the same initial distances. But Theorem 14 proves the existence of hyperbolic relative equilibria in \mathbf{H}^2 for three equal masses. The bodies move along hyperbolas of the hyperboloid that models \mathbf{H}^2 remaining all the time on a moving geodesic and maintaining the initial distances among themselves. These orbits rather resemble fighter planes flying in formation than celestial bodies moving under the action of gravity alone. The result also holds if the mass in the middle differs from the other two. The last result of this section, Theorem 15, shows that parabolic relative equilibria do not exist.

2.4. Saari's conjecture. Our extension of the Newtonian n -body problem to spaces of constant curvature also reveals new aspects of Saari's conjecture. Proposed in 1970 by Don Saari in the Euclidean case, Saari's conjecture claims that solutions with constant moment of inertia are relative equilibria. This problem generated a lot of interest from the very beginning, but also several failed attempts to prove it. The discovery of the figure eight solution, which has an almost constant moment of inertia, and whose existence was proved in 2000 by Alain Chenciner and Richard Montgomery, [14], renewed the interest in this conjecture. Several results showed up not long thereafter. The case $n = 3$ was solved in 2005 by Rick Moeckel, [56]; the collinear case, for any number of bodies and the more general potentials that involve only mutual distances, was settled the same year by the authors of this paper, [25]. Saari's conjecture is also connected to the Chazy-Wintner-Smale conjecture, [70], [76], which asks to determine whether the number of central configurations is finite for n given bodies in Euclidean space.

Since relative equilibria have elliptic and hyperbolic versions in \mathbf{H}^2 , Saari's conjecture raises new questions for $\kappa < 0$. We answered them in Theorem 16 of Section 7, when the bodies are restrained to a geodesic that rotates elliptically or hyperbolically.

An Appendix in which we present some basic facts about the Weierstrass model of the hyperbolic plane, together with some historical remarks, closes our paper. We suggest that readers unfamiliar with this model take a look at the Appendix before getting into the technical details related to our results.

2.5. Some physical remarks. Does our gravitational model have any connection with the physical reality? Since there is no unique extension of the Newtonian n -body problem to spaces of constant curvature, is our generalization meaningful from the physical point of view or does it lead only to some interesting mathematical properties?

We followed the tradition of the cotangent potential, which seems the most natural candidate. But since the debate on the nature of the physical space is open, the only way to justify this model is through mathematical results. As we will further argue, not only that the properties we obtained match the Euclidean ones, but they also provide a classical explanation of the cosmological scenario, in agreement with the basic conclusions of general relativity.

But before getting into the physical aspect, let us remark that our model is based on mathematical principles, which lead to a meaningful physical interpretation. As we already mentioned, the cotangent potential preserves two fundamental properties: (i) it is harmonic for the one-body problem and (ii) it generates a central field in which all bounded orbits are closed. Other results that support the cotangent potential are based on the idea of central (or gnomonic) projection, [1]. By taking the central projection on the sphere for the planar Kepler problem, Paul Appell obtained the cotangent potential. This idea can be generalized by projecting the planar Kepler problem to any surface of revolution, as one of us (Manuele Santoprete) proved, [61].

In 1992, Kozlov and Harin showed that the only central potential that satisfies the fundamental properties (i) and (ii) in \mathbf{S}^2 and has meaning in celestial mechanics is the cotangent of the distance, [43]. This fact had been known to Infeld for the quantum mechanical version of the potential, [36]. But since any continuously differentiable and non-constant harmonic function attains no maximum or minimum on the sphere, the existence of two distinct singularities (the collisional and the antipodal—in our case) is not unexpected. And though a force that becomes infinite for points at opposite poles may seem counterintuitive in a gravitational framework, it explains the cosmological scenario.

Indeed, while there is no doubt that n point masses ejecting from a total collapse would move forever in spaces with $\kappa \leq 0$ for large initial conditions, in agreement with general relativity, it is not clear what happens for $\kappa > 0$. But the energy relation (22) shows that, in spherical space, the current expansion of the universe cannot last forever. For a fixed energy constant, h , the potential energy, $-U$, would become positive and very large if one or more pairs of particles were to come close to antipodal singularities. Therefore in a homogeneous universe, highly populated with non-colliding particles, the system could never expand beyond the equator (assuming that the initial ejection took place at one pole). No matter how large (but fixed) the energy constant is, when the potential energy reaches the value h , the kinetic energy becomes zero, so all the particles stop simultaneously and the motion reverses.

Thus, for $\kappa > 0$, the cotangent potential recovers the growth of the system to a maximum size and the reversal of the expansion independently on the value of the energy constant. Without antipodal singularities, the reversal could take place only for certain initial conditions. This conclusion is reached without introducing a cosmological force and differently from how it was obtained in the classical model proposed by Élie Cartan, [11], [12], and shown by Frank Tipler to be as rigorous as Friedmann's cosmology, [72], [73].

Another result that suggests the validity of the cotangent potential is the nonexistence of fixed points. They don't show up in the Euclidean case, and neither do they appear in this model within the observable universe. The properties we proved for relative equilibria are also in agreement with the classical n -body problem, the only exception being the Lagrangian solutions for $\kappa \neq 0$, which, unlike in the Euclidean case, must have equal masses and move on the same circle. This distinction adds to the strength of the model because, even in the Euclidean case, the arbitrariness of the Lagrangian solutions is a peculiar property. At least two arguments support this point of view. First, relative equilibria generated from regular polygons, except the equilateral triangle, exist only if the masses are equal. The second argument is related to central configurations, which generate relative equilibria in the Euclidean case. One of us (Florin Diacu) proved that among attraction forces given by symmetric laws of masses, $\gamma(m_i, m_j) = \gamma(m_j, m_i)$, equilateral central configurations with unequal masses occur only when $\gamma(m_i, m_j) = c m_i m_j$, where c is a nonzero constant, [20]. Since for $\kappa \neq 0$ relative equilibria are equilateral only if the masses are equal means that Lagrangian solutions of arbitrary masses characterize the Euclidean space.

Such orbits exist in nature, the best known example being the equilateral triangle formed by the Sun, Jupiter, and the Trojan asteroids. Therefore our result reinforces the fact that space is Euclidean within distances comparable to those of our solar system. This fact was not known during the time of Gauss, who apparently tried to determine the nature of space by measuring the angles of triangles having the vertices some tens of kilometers apart.⁷ Since we cannot measure the angles of cosmic triangles, our result opens up a new possibility. Any evidence of a Lagrangian solution involving galaxies (or clusters of galaxies) of unequal masses, could be used as an argument for the flatness of the physical space for distances comparable to the size of the triangle. Similarly, hyperbolic relative equilibria would show that space has negative curvature.

⁷Arthur Miller argues that these experiments never took place, [55]

3. EQUATIONS OF MOTION

We derive in this section a Newtonian n -body problem on surfaces of constant curvature. The equations of motion we obtain are simple enough to allow an analytic approach. At the end, we provide a straightforward generalization of these equations to spaces of constant curvature of any finite dimension.

3.1. Unified trigonometry. Let us first consider what, following [9], we will call trigonometric κ -functions, which unify elliptical and hyperbolic trigonometry. We define the κ -sine, sn_κ , as

$$\text{sn}_\kappa(x) := \begin{cases} \kappa^{-1/2} \sin \kappa^{1/2} x & \text{if } \kappa > 0 \\ x & \text{if } \kappa = 0 \\ (-\kappa)^{-1/2} \sinh(-\kappa)^{1/2} x & \text{if } \kappa < 0, \end{cases}$$

the κ -cosine, csn_κ , as

$$\text{csn}_\kappa(x) := \begin{cases} \cos \kappa^{1/2} x & \text{if } \kappa > 0 \\ 1 & \text{if } \kappa = 0 \\ \cosh(-\kappa)^{1/2} x & \text{if } \kappa < 0, \end{cases}$$

as well as the κ -tangent, tn_κ , and κ -cotangent, ctn_κ , as

$$\text{tn}_\kappa(x) := \frac{\text{sn}_\kappa(x)}{\text{csn}_\kappa(x)} \quad \text{and} \quad \text{ctn}_\kappa(x) := \frac{\text{csn}_\kappa(x)}{\text{sn}_\kappa(x)},$$

respectively. The entire trigonometry can be rewritten in this unified context, but the only identity we will further need is the fundamental formula

$$\kappa \text{sn}_\kappa^2(x) + \text{csn}_\kappa^2(x) = 1.$$

3.2. Differential-geometric approach. In a 2-dimensional Riemann space, we can define geodesic polar coordinates, (r, ϕ) , by fixing an origin and an oriented geodesic through it. If the space has constant curvature κ , the range of r depends on κ ; namely $r \in [0, \pi/(2\kappa^{1/2})]$ for $\kappa > 0$ and $r \in [0, \infty)$ for $\kappa \leq 0$; in all cases, $\phi \in [0, 2\pi]$. The line element is given by

$$ds_\kappa^2 = dr^2 + \text{sn}_\kappa^2(r) d\phi^2.$$

In \mathbf{S}^2 , \mathbf{R}^2 , and \mathbf{H}^2 , the line element corresponds to $\kappa = 1, 0$, and -1 , respectively, and reduces therefore to

$$ds_1^2 = dr^2 + (\sin^2 r) d\phi^2, \quad ds_0^2 = dr^2 + r^2 d\phi^2, \quad \text{and} \quad ds_{-1}^2 = dr^2 + (\sinh^2 r) d\phi^2.$$

In [9], the Lagrangian of the Kepler problem is defined as

$$L_\kappa(r, \phi, v_r, v_\phi) = \frac{1}{2} [v_r^2 + \text{sn}_\kappa^2(r) v_\phi^2] + U_\kappa(r),$$

where v_r and v_ϕ represent the polar components of the velocity, and $-U$ is the potential, where

$$U_\kappa(r) = G \text{ctn}_\kappa(r)$$

is the force function, $G > 0$ being the gravitational constant. This means that the corresponding force functions in \mathbf{S}^2 , \mathbf{R}^2 , and \mathbf{H}^2 are, respectively,

$$U_1(r) = G \cot r, \quad U_0(r) = Gr^{-1}, \quad \text{and} \quad U_{-1}(r) = G \coth r.$$

In this setting, the case $\kappa = 0$ separates the potentials with $\kappa > 0$ and $\kappa < 0$ into classes exhibiting different qualitative behavior. The passage from $\kappa > 0$ to $\kappa < 0$ through $\kappa = 0$ takes place continuously. Moreover, the potential is spherically symmetric and satisfies Gauss's law in a 3-dimensional space of constant curvature κ . This law asks that the flux of the radial force field across a sphere of radius r is a constant independent of r . Since the area of the sphere is $4\pi sn_k^2(r)$, the flux is $4\pi sn_k^2(r) \times \frac{d}{dr}U_\kappa(r)$, so the potential satisfies Gauss's law. As in the Euclidean case, this generalized potential does not satisfy Gauss's law in the 2-dimensional space. The results obtained in [9] show that the force function U_κ leads to the expected conic orbits on surfaces of constant curvature, and thus justify this extension of the Kepler problem to $\kappa \neq 0$.

3.3. The potential. To generalize the above setting of the Kepler problem to the n -body problem on surfaces of constant curvature, let us start with some notations. Consider n bodies of masses m_1, \dots, m_n moving on a surface of constant curvature κ . When $\kappa > 0$, the surfaces are spheres of radii $\kappa^{-1/2}$ given by the equation $x^2 + y^2 + z^2 = \kappa^{-1}$; for $\kappa = 0$, we recover the Euclidean plane; and if $\kappa < 0$, we consider the Weierstrass model of hyperbolic geometry (see Appendix), which is devised on the sheets with $z > 0$ of the hyperboloids of two sheets $x^2 + y^2 - z^2 = \kappa^{-1}$. The coordinates of the body of mass m_i are given by $\mathbf{q}_i = (x_i, y_i, z_i)$ and a constraint, depending on κ , that restricts the motion of this body to one of the above described surfaces.

In this paper, $\tilde{\nabla}_{\mathbf{q}_i}$ denotes either of the gradient operators

$$\nabla_{\mathbf{q}_i} = (\partial_{x_i}, \partial_{y_i}, \partial_{z_i}), \text{ for } \kappa \geq 0, \quad \text{or} \quad \bar{\nabla}_{\mathbf{q}_i} = (\partial_{x_i}, \partial_{y_i}, -\partial_{z_i}), \text{ for } \kappa < 0,$$

with respect to the vector \mathbf{q}_i , and $\tilde{\nabla}$ stands for the operator $(\tilde{\nabla}_{\mathbf{q}_1}, \dots, \tilde{\nabla}_{\mathbf{q}_n})$. For $\mathbf{a} = (a_x, a_y, a_z)$ and $\mathbf{b} = (b_x, b_y, b_z)$ in \mathbf{R}^3 , we define $\mathbf{a} \odot \mathbf{b}$ as either of the inner products

$$\mathbf{a} \cdot \mathbf{b} := (a_x b_x + a_y b_y + a_z b_z) \quad \text{for } \kappa \geq 0,$$

$$\mathbf{a} \square \mathbf{b} := (a_x b_x + a_y b_y - a_z b_z) \quad \text{for } \kappa < 0,$$

the latter being the Lorentz inner product (see Appendix). We also define $\mathbf{a} \otimes \mathbf{b}$ as either of the cross products

$$\mathbf{a} \times \mathbf{b} := (a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x) \quad \text{for } \kappa \geq 0,$$

$$\mathbf{a} \boxtimes \mathbf{b} := (a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_y b_x - a_x b_y) \quad \text{for } \kappa < 0.$$

The distance between \mathbf{a} and \mathbf{b} on the surface of constant curvature κ is then given by

$$d_\kappa(\mathbf{a}, \mathbf{b}) := \begin{cases} \kappa^{-1/2} \cos^{-1}(\kappa \mathbf{a} \cdot \mathbf{b}), & \kappa > 0 \\ |\mathbf{a} - \mathbf{b}|, & \kappa = 0 \\ (-\kappa)^{-1/2} \cosh^{-1}(\kappa \mathbf{a} \boxminus \mathbf{b}), & \kappa < 0, \end{cases}$$

where the vertical bars denote the standard Euclidean norm. In particular, the distances in \mathbf{S}^2 and \mathbf{H}^2 are

$$d_1(\mathbf{a}, \mathbf{b}) = \cos^{-1}(\mathbf{a} \cdot \mathbf{b}), \quad d_{-1}(\mathbf{a}, \mathbf{b}) = \cosh^{-1}(-\mathbf{a} \boxminus \mathbf{b}),$$

respectively. Notice that d_0 is the limiting case of d_κ when $\kappa \rightarrow 0$. Indeed, for both $\kappa > 0$ and $\kappa < 0$, the vectors \mathbf{a} and \mathbf{b} tend to infinity and become parallel, while the surfaces tend to an Euclidean plane, therefore the length of the arc between the vectors tends to the Euclidean distance.

We will further define a potential in \mathbf{R}^3 if $\kappa > 0$, and in the 3-dimensional Minkowski space \mathcal{M}^3 (see Appendix) if $\kappa < 0$, such that we can use a variational method to derive the equations of motion. For this purpose we need to extend the distance to these spaces. We do this by redefining the distance as

$$d_\kappa(\mathbf{a}, \mathbf{b}) := \begin{cases} \kappa^{-1/2} \cos^{-1} \frac{\kappa \mathbf{a} \cdot \mathbf{b}}{\sqrt{\kappa \mathbf{a} \cdot \mathbf{a}} \sqrt{\kappa \mathbf{b} \cdot \mathbf{b}}}, & \kappa > 0 \\ |\mathbf{a} - \mathbf{b}|, & \kappa = 0 \\ (-\kappa)^{-1/2} \cosh^{-1} \frac{\kappa \mathbf{a} \boxminus \mathbf{b}}{\sqrt{\kappa \mathbf{a} \boxminus \mathbf{a}} \sqrt{\kappa \mathbf{b} \boxminus \mathbf{b}}}, & \kappa < 0. \end{cases}$$

Notice that this new definition is identical with the previous one when we restrict the vectors \mathbf{a} and \mathbf{b} to the spheres $x^2 + y^2 + z^2 = \kappa^{-1}$ or the hyperboloids $x^2 + y^2 - z^2 = \kappa^{-1}$, but is also valid for any vectors \mathbf{a} and \mathbf{b} in \mathbf{R}^3 and \mathcal{M}^3 , respectively.

From now on we will rescale the units such that the gravitational constant G is 1. We thus define the potential of the n -body problem as the function $-U_\kappa(\mathbf{q})$, where

$$U_\kappa(\mathbf{q}) := \frac{1}{2} \sum_{i=1}^n \sum_{j=1, j \neq i}^n m_i m_j \text{ctn}_\kappa(d_\kappa(\mathbf{q}_i, \mathbf{q}_j))$$

stands for the force function, and $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_n)$ is the configuration of the system. Notice that $\text{ctn}_0(d_0(\mathbf{q}_i, \mathbf{q}_j)) = |\mathbf{q}_i - \mathbf{q}_j|^{-1}$, which means that we recover the Newtonian potential in the Euclidean case. Therefore the potential U_κ varies continuously with the curvature κ .

Now that we defined a potential that satisfies the basic continuity condition we required of any extension of the n -body problem beyond the Euclidean space, we will focus on the case $\kappa \neq 0$. A straightforward computation shows

that

$$(1) \quad U_\kappa(\mathbf{q}) = \frac{1}{2} \sum_{i=1}^n \sum_{j=1, j \neq i}^n \frac{m_i m_j (\sigma \kappa)^{1/2} \frac{\kappa \mathbf{q}_i \odot \mathbf{q}_j}{\sqrt{\kappa \mathbf{q}_i \odot \mathbf{q}_i} \sqrt{\kappa \mathbf{q}_j \odot \mathbf{q}_j}}}{\sqrt{\sigma - \sigma \left(\frac{\kappa \mathbf{q}_i \odot \mathbf{q}_j}{\sqrt{\kappa \mathbf{q}_i \odot \mathbf{q}_i} \sqrt{\kappa \mathbf{q}_j \odot \mathbf{q}_j}} \right)^2}}, \quad \kappa \neq 0,$$

where

$$\sigma = \begin{cases} +1, & \text{for } \kappa > 0 \\ -1, & \text{for } \kappa < 0. \end{cases}$$

3.4. Euler's formula. Notice that $U_\kappa(\eta \mathbf{q}) = U_\kappa(\mathbf{q}) = \eta^0 U_\kappa(\mathbf{q})$ for any $\eta \neq 0$, which means that the potential is a homogeneous function of degree zero. But for \mathbf{q} in \mathbf{R}^{3n} , homogeneous functions $F : \mathbf{R}^{3n} \rightarrow \mathbf{R}$ of degree α satisfy Euler's formula, $\mathbf{q} \cdot \nabla F(\mathbf{q}) = \alpha F(\mathbf{q})$. With our notations, Euler's formula can be written as $\mathbf{q} \odot \tilde{\nabla} F(\mathbf{q}) = \alpha F(\mathbf{q})$. Since $\alpha = 0$ for U_κ with $\kappa \neq 0$, we conclude that

$$(2) \quad \mathbf{q} \odot \tilde{\nabla} U_\kappa(\mathbf{q}) = 0.$$

We can also write the force function as $U_\kappa(\mathbf{q}) = \frac{1}{2} \sum_{i=1}^n U_\kappa^i(\mathbf{q}_i)$, where

$$U_\kappa^i(\mathbf{q}_i) := \sum_{j=1, j \neq i}^n \frac{m_i m_j (\sigma \kappa)^{1/2} \frac{\kappa \mathbf{q}_i \odot \mathbf{q}_j}{\sqrt{\kappa \mathbf{q}_i \odot \mathbf{q}_i} \sqrt{\kappa \mathbf{q}_j \odot \mathbf{q}_j}}}{\sqrt{\sigma - \sigma \left(\frac{\kappa \mathbf{q}_i \odot \mathbf{q}_j}{\sqrt{\kappa \mathbf{q}_i \odot \mathbf{q}_i} \sqrt{\kappa \mathbf{q}_j \odot \mathbf{q}_j}} \right)^2}}, \quad i = 1, \dots, n,$$

are also homogeneous functions of degree 0. Applying Euler's formula for functions $F : \mathbf{R}^3 \rightarrow \mathbf{R}$, we obtain that $\mathbf{q}_i \odot \tilde{\nabla}_{\mathbf{q}_i} U_\kappa^i(\mathbf{q}) = 0$. Then using the identity $\tilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = \tilde{\nabla}_{\mathbf{q}_i} U_\kappa^i(\mathbf{q}_i)$, we can conclude that

$$(3) \quad \mathbf{q}_i \odot \tilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = 0, \quad i = 1, \dots, n.$$

3.5. Derivation of the equations of motion. To obtain the equations of motion for $\kappa \neq 0$, we will use a variational method applied to the force function (1). The Lagrangian of the n -body system has the form

$$L_\kappa(\mathbf{q}, \dot{\mathbf{q}}) = T_\kappa(\mathbf{q}, \dot{\mathbf{q}}) + U_\kappa(\mathbf{q}),$$

where $T_\kappa(\mathbf{q}, \dot{\mathbf{q}}) := \frac{1}{2} \sum_{i=1}^n m_i (\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i) (\kappa \mathbf{q}_i \odot \mathbf{q}_i)$ is the kinetic energy of the system. (The reason for introducing the factors $\kappa \mathbf{q}_i \odot \mathbf{q}_i = 1$ into the definition of the kinetic energy will become clear in Section 3.8.) Then, according to the theory of constrained Lagrangian dynamics (see, e.g., [32]), the equations of motion are

$$(4) \quad \frac{d}{dt} \left(\frac{\partial L_\kappa}{\partial \dot{\mathbf{q}}_i} \right) - \frac{\partial L_\kappa}{\partial \mathbf{q}_i} - \lambda_\kappa^i(t) \frac{\partial f_i}{\partial \mathbf{q}_i} = \mathbf{0}, \quad i = 1, \dots, n,$$

where $f_\kappa^i = \mathbf{q}_i \odot \mathbf{q}_i - \kappa^{-1}$ is the function that gives the constraint $f_\kappa^i = 0$, which keeps the body of mass m_i on the surface of constant curvature κ , and λ_κ^i is the Lagrange multiplier corresponding to the same body. Since $\mathbf{q}_i \odot \mathbf{q}_i = \kappa^{-1}$ implies that $\dot{\mathbf{q}}_i \odot \mathbf{q}_i = 0$, it follows that

$$\frac{d}{dt} \left(\frac{\partial L_\kappa}{\partial \dot{\mathbf{q}}_i} \right) = m_i \ddot{\mathbf{q}}_i (\kappa \mathbf{q}_i \odot \mathbf{q}_i) + 2m_i (\kappa \dot{\mathbf{q}}_i \odot \mathbf{q}_i) = m_i \ddot{\mathbf{q}}_i.$$

This relation, together with

$$\frac{\partial L_\kappa}{\partial \mathbf{q}_i} = m_i \kappa (\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i) \mathbf{q}_i + \tilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q}),$$

implies that equations (4) are equivalent to

$$(5) \quad m_i \ddot{\mathbf{q}}_i - m_i \kappa (\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i) \mathbf{q}_i - \tilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q}) - 2\lambda_\kappa^i(t) \mathbf{q}_i = \mathbf{0}, \quad i = 1, \dots, n.$$

To determine λ_κ^i , notice that $0 = \ddot{f}_\kappa^i = 2\ddot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i + 2(\mathbf{q}_i \odot \ddot{\mathbf{q}}_i)$, so

$$(6) \quad \mathbf{q}_i \odot \ddot{\mathbf{q}}_i = -\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i.$$

Let us also remark that \odot -multiplying equations (5) by \mathbf{q}_i and using (3), we obtain that

$$m_i (\mathbf{q}_i \odot \ddot{\mathbf{q}}_i) - m_i \kappa (\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i) - \mathbf{q}_i \odot \tilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = 2\lambda_\kappa^i \mathbf{q}_i \odot \mathbf{q}_i = 2\kappa^{-1} \lambda_\kappa^i,$$

which, via (6), implies that $\lambda_\kappa^i = -\kappa m_i (\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i)$. Substituting these values of the Lagrange multipliers into equations (5), the equations of motion and their constraints become

$$(7) \quad m_i \ddot{\mathbf{q}}_i = \tilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q}) - m_i \kappa (\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i) \mathbf{q}_i, \quad \mathbf{q}_i \odot \mathbf{q}_i = \kappa^{-1}, \quad \kappa \neq 0, \\ i = 1, \dots, n.$$

The \mathbf{q}_i -gradient of the force function, obtained from (1), has the form

$$(8) \quad \tilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = \sum_{j=1, j \neq i}^n \frac{\frac{m_i m_j (\sigma \kappa)^{1/2} \left(\sigma \kappa \mathbf{q}_j - \sigma \frac{\kappa^2 \mathbf{q}_i \odot \mathbf{q}_j}{\kappa \mathbf{q}_i \odot \mathbf{q}_i} \mathbf{q}_i \right)}{\sqrt{\kappa \mathbf{q}_i \odot \mathbf{q}_i} \sqrt{\kappa \mathbf{q}_j \odot \mathbf{q}_j}}}{\left[\sigma - \sigma \left(\frac{\kappa \mathbf{q}_i \odot \mathbf{q}_j}{\sqrt{\kappa \mathbf{q}_i \odot \mathbf{q}_i} \sqrt{\kappa \mathbf{q}_j \odot \mathbf{q}_j}} \right)^2 \right]^{3/2}}, \quad \kappa \neq 0,$$

and using the fact that $\kappa \mathbf{q}_i \odot \mathbf{q}_i = 1$, we can write this gradient as

$$(9) \quad \tilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = \sum_{j=1, j \neq i}^n \frac{m_i m_j (\sigma \kappa)^{3/2} [\mathbf{q}_j - (\kappa \mathbf{q}_i \odot \mathbf{q}_j) \mathbf{q}_i]}{[\sigma - \sigma (\kappa \mathbf{q}_i \odot \mathbf{q}_j)^2]^{3/2}}, \quad \kappa \neq 0.$$

Sometimes we can use the simpler form (9) of the gradient, but whenever we need to exploit the homogeneity of the gradient or have to differentiate it, we must use its original form (8). Thus equations (7) and (8) describe the n -body problem on surfaces of constant curvature for $\kappa \neq 0$. Though more

complicated than the equations of motion Newton derived for the Euclidean space, system (7) is simple enough to allow an analytic approach. Let us first provide some of its basic properties.

3.6. First integrals. The equations of motion have the energy integral

$$(10) \quad T_\kappa(\mathbf{q}, \mathbf{p}) - U_\kappa(\mathbf{q}) = h,$$

where, recall, $T_\kappa(\mathbf{q}, \mathbf{p}) := \frac{1}{2} \sum_{i=1}^n m_i^{-1} (\mathbf{p}_i \odot \mathbf{p}_i) (\kappa \mathbf{q}_i \odot \mathbf{q}_i)$ is the kinetic energy, $\mathbf{p} := (\mathbf{p}_1, \dots, \mathbf{p}_n)$ denotes the momentum of the n -body system, with $\mathbf{p}_i := m_i \dot{\mathbf{q}}_i$ representing the momentum of the body of mass m_i , $i = 1, \dots, n$, and h is a real constant. Indeed, \odot -multiplying equations (7) by $\dot{\mathbf{q}}_i$, we obtain

$$\sum_{i=1}^n m_i \ddot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i = [\tilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q})] \odot \dot{\mathbf{q}}_i - \sum_{i=1}^n m_i \kappa (\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i) \mathbf{q}_i \odot \dot{\mathbf{q}}_i = \frac{d}{dt} U_\kappa(\mathbf{q}(t)).$$

Then equation (10) follows by integrating the first and last term in the above equation.

The equations of motion also have the integrals of the angular momentum,

$$(11) \quad \sum_{i=1}^n \mathbf{q}_i \otimes \mathbf{p}_i = \mathbf{c},$$

where \mathbf{c} is a constant vector. Relations (11) follow by integrating the identity formed by the first and last term of the equations

$$(12) \quad \sum_{i=1}^n m_i \ddot{\mathbf{q}}_i \otimes \mathbf{q}_i = \sum_{i=1}^n \sum_{j=1, j \neq i}^n \frac{m_i m_j (\sigma \kappa)^{3/2} \mathbf{q}_i \otimes \mathbf{q}_j}{[\sigma - \sigma(\kappa \mathbf{q}_i \odot \mathbf{q}_j)^2]^{3/2}} \\ - \sum_{i=1}^n \left[\sum_{j=1, j \neq i}^n \frac{m_i m_j (\sigma \kappa)^{3/2} (\kappa \mathbf{q}_i \odot \mathbf{q}_j)}{[\sigma - \sigma(\kappa \mathbf{q}_i \odot \mathbf{q}_j)^2]^{3/2}} - m_i \kappa (\dot{\mathbf{q}}_i \odot \dot{\mathbf{q}}_i) \right] \mathbf{q}_i \otimes \mathbf{q}_i = \mathbf{0},$$

obtained if \otimes -multiplying the equations of motion (7) by \mathbf{q}_i . The last of the above identities follows from the skew-symmetry of \otimes and the fact that $\mathbf{q}_i \otimes \mathbf{q}_i = \mathbf{0}$, $i = 1, \dots, n$.

3.7. Motion of a free body. A consequence of the integrals of motion is the analogue of the well known result from the Euclidean space related to the motion of a single body in the absence of any gravitational interactions. Though simple, the proof of this property is not as trivial as in the classical case.

Proposition 1. *A free body on a surface of constant curvature is either at rest or it moves uniformly along a geodesic. Moreover, for $\kappa > 0$, every orbit is closed.*

Proof. Since there are no gravitational interactions, the equations of motion take the form

$$(13) \quad \ddot{\mathbf{q}} = -\kappa(\dot{\mathbf{q}} \odot \dot{\mathbf{q}})\mathbf{q},$$

where $\mathbf{q} = (x, y, z)$ is the vector describing the position of the body of mass m . If $\dot{\mathbf{q}}(0) = \mathbf{0}$, then $\ddot{\mathbf{q}}(0) = \mathbf{0}$, so no force acts on m . Therefore the body will be at rest.

If $\dot{\mathbf{q}}(0) \neq \mathbf{0}$, $\ddot{\mathbf{q}}(0)$ and $\mathbf{q}(0)$ are collinear, having the same sense if $\kappa < 0$, but the opposite sense if $\kappa > 0$. So the sum between $\ddot{\mathbf{q}}(0)$ and $\dot{\mathbf{q}}(0)$ pulls the body along the geodesic corresponding to the direction of these vectors.

We still need to show that the motion is uniform. This fact follows obviously from the integral of energy, but we can also derive it from the integrals of the angular momentum. Indeed, for $\kappa > 0$, these integrals lead us to

$$c = (\mathbf{q} \times \dot{\mathbf{q}}) \cdot (\mathbf{q} \times \dot{\mathbf{q}}) = (\mathbf{q} \cdot \mathbf{q})(\dot{\mathbf{q}} \cdot \dot{\mathbf{q}}) \sin^2 \alpha,$$

where c is the length of the angular momentum vector and α is the angle between \mathbf{q} and $\dot{\mathbf{q}}$ (namely $\pi/2$). So since $\mathbf{q} \cdot \mathbf{q} = \kappa^{-1}$, we can draw the conclusion that the speed of the body is constant.

For $\kappa < 0$, we can write that

$$c = (\mathbf{q} \boxtimes \dot{\mathbf{q}}) \boxtimes (\mathbf{q} \boxtimes \dot{\mathbf{q}}) = - \begin{vmatrix} \mathbf{q} \boxtimes \mathbf{q} & \mathbf{q} \boxtimes \dot{\mathbf{q}} \\ \mathbf{q} \boxtimes \dot{\mathbf{q}} & \dot{\mathbf{q}} \boxtimes \dot{\mathbf{q}} \end{vmatrix} = - \begin{vmatrix} \kappa^{-1} & 0 \\ 0 & \dot{\mathbf{q}} \boxtimes \dot{\mathbf{q}} \end{vmatrix} = -\kappa^{-1} \dot{\mathbf{q}} \boxtimes \dot{\mathbf{q}}.$$

Therefore the speed is constant in this case too, so the motion is uniform. Since for $\kappa > 0$ the body moves on geodesics of a sphere, every orbit is closed. \square

3.8. Hamiltonian form. The equations of motion (7) are Hamiltonian. Indeed, the Hamiltonian function H_κ is given by

$$\begin{cases} H_\kappa(\mathbf{q}, \mathbf{p}) = \frac{1}{2} \sum_{i=1}^n m_i^{-1} (\mathbf{p}_i \odot \mathbf{p}_i) (\kappa \mathbf{q}_i \odot \mathbf{q}_i) - U_\kappa(\mathbf{q}), \\ \mathbf{q}_i \odot \mathbf{q}_i = \kappa^{-1}, \quad \kappa \neq 0, \quad i = 1, \dots, n. \end{cases}$$

Equations (5) thus take the form of a $6n$ -dimensional first order system of differential equations with $2n$ constraints,

$$(14) \quad \begin{cases} \dot{\mathbf{q}}_i = \tilde{\nabla}_{\mathbf{p}_i} H_\kappa(\mathbf{q}, \mathbf{p}) = m_i^{-1} \mathbf{p}_i, \\ \dot{\mathbf{p}}_i = -\tilde{\nabla}_{\mathbf{q}_i} H_\kappa(\mathbf{q}, \mathbf{p}) = \tilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q}) - m_i^{-1} \kappa (\mathbf{p}_i \odot \mathbf{p}_i) \mathbf{q}_i, \\ \mathbf{q}_i \odot \mathbf{q}_i = \kappa^{-1}, \quad \mathbf{q}_i \odot \mathbf{p}_i = 0, \quad \kappa \neq 0, \quad i = 1, \dots, n. \end{cases}$$

It is interesting to note that, independently of whether the kinetic energy is defined as

$$T_\kappa(\mathbf{p}) := \frac{1}{2} \sum_{i=1}^n m_i^{-1} \mathbf{p}_i \odot \mathbf{p}_i \quad \text{or} \quad T_\kappa(\mathbf{q}, \mathbf{p}) := \frac{1}{2} \sum_{i=1}^n m_i^{-1} (\mathbf{p}_i \odot \mathbf{p}_i) (\kappa \mathbf{q}_i \odot \mathbf{q}_i),$$

(which, though identical since $\kappa \mathbf{q}_i \odot \mathbf{q}_i = 1$, does not come to the same thing when differentiating T_κ), the form of equations (7) remains the same. But

in the former case, system (7) cannot be put in Hamiltonian form in spite of having an energy integral, while in the former case it can. This is why we chose the latter definition of T_κ .

These equations describe the motion of the n -body system for any $\kappa \neq 0$, the case $\kappa = 0$ corresponding to the classical Newtonian equations. The representative non-zero-curvature cases, however, are $\kappa = 1$ and $\kappa = -1$, which characterize the motion for $\kappa > 0$ and $\kappa < 0$, respectively. Therefore we will further focus on the n -body problem in \mathbf{S}^2 and \mathbf{H}^2 .

3.9. Equations of motion in \mathbf{S}^2 . In this case, the force function (1) takes the form

$$(15) \quad U_1(\mathbf{q}) = \frac{1}{2} \sum_{i=1}^n \sum_{j=1, j \neq i}^n \frac{m_i m_j \frac{\mathbf{q}_i \cdot \mathbf{q}_j}{\sqrt{\mathbf{q}_i \cdot \mathbf{q}_i} \sqrt{\mathbf{q}_j \cdot \mathbf{q}_j}}}{\sqrt{1 - \left(\frac{\mathbf{q}_i \cdot \mathbf{q}_j}{\sqrt{\mathbf{q}_i \cdot \mathbf{q}_i} \sqrt{\mathbf{q}_j \cdot \mathbf{q}_j}} \right)^2}},$$

while the equations of motion (7) and their constraints become

$$(16) \quad m_i \ddot{\mathbf{q}}_i = \nabla_{\mathbf{q}_i} U_1(\mathbf{q}) - m_i (\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i) \mathbf{q}_i, \quad \mathbf{q}_i \cdot \mathbf{q}_i = 1, \quad \mathbf{q}_i \cdot \dot{\mathbf{q}}_i = 0, \quad i = 1, \dots, n.$$

In terms of coordinates, the equations of motion and their constraints can be written as

$$(17) \quad \begin{cases} m_i \ddot{x}_i = \frac{\partial U_1}{\partial x_i} - m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) x_i, \\ m_i \ddot{y}_i = \frac{\partial U_1}{\partial y_i} - m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) y_i, \\ m_i \ddot{z}_i = \frac{\partial U_1}{\partial z_i} - m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) z_i, \\ x_i^2 + y_i^2 + z_i^2 = 1, \quad x_i \dot{x}_i + y_i \dot{y}_i + z_i \dot{z}_i = 0, \quad i = 1, \dots, n, \end{cases}$$

and by computing the gradients they become

$$(18) \quad \begin{cases} \ddot{x}_i = \sum_{j=1, j \neq i}^n \frac{m_j \frac{x_j - \frac{x_i x_j + y_i y_j + z_i z_j}{x_i^2 + y_i^2 + z_i^2} x_i}{\sqrt{x_i^2 + y_i^2 + z_i^2} \sqrt{x_j^2 + y_j^2 + z_j^2}}}{\left[1 - \left(\frac{x_i x_j + y_i y_j + z_i z_j}{\sqrt{x_i^2 + y_i^2 + z_i^2} \sqrt{x_j^2 + y_j^2 + z_j^2}} \right)^2 \right]^{3/2}} - (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) x_i, \\ \ddot{y}_i = \sum_{j=1, j \neq i}^n \frac{m_j \frac{y_j - \frac{x_i x_j + y_i y_j + z_i z_j}{x_i^2 + y_i^2 + z_i^2} y_i}{\sqrt{x_i^2 + y_i^2 + z_i^2} \sqrt{x_j^2 + y_j^2 + z_j^2}}}{\left[1 - \left(\frac{x_i x_j + y_i y_j + z_i z_j}{\sqrt{x_i^2 + y_i^2 + z_i^2} \sqrt{x_j^2 + y_j^2 + z_j^2}} \right)^2 \right]^{3/2}} - (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) y_i, \\ \ddot{z}_i = \sum_{j=1, j \neq i}^n \frac{m_j \frac{z_j - \frac{x_i x_j + y_i y_j + z_i z_j}{x_i^2 + y_i^2 + z_i^2} z_i}{\sqrt{x_i^2 + y_i^2 + z_i^2} \sqrt{x_j^2 + y_j^2 + z_j^2}}}{\left[1 - \left(\frac{x_i x_j + y_i y_j + z_i z_j}{\sqrt{x_i^2 + y_i^2 + z_i^2} \sqrt{x_j^2 + y_j^2 + z_j^2}} \right)^2 \right]^{3/2}} - (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) z_i, \\ x_i^2 + y_i^2 + z_i^2 = 1, \quad x_i \dot{x}_i + y_i \dot{y}_i + z_i \dot{z}_i = 0, \quad i = 1, \dots, n. \end{cases}$$

Since we will neither need the homogeneity of the gradient, nor we will we differentiate it, we can use the constraints to write the above system as

$$(19) \quad \begin{cases} \ddot{x}_i = \sum_{j=1, j \neq i}^n \frac{m_j [x_j - (x_i x_j + y_i y_j + z_i z_j) x_i]}{[1 - (x_i x_j + y_i y_j + z_i z_j)^2]^{3/2}} - (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) x_i, \\ \ddot{y}_i = \sum_{j=1, j \neq i}^n \frac{m_j [y_j - (x_i x_j + y_i y_j + z_i z_j) y_i]}{[1 - (x_i x_j + y_i y_j + z_i z_j)^2]^{3/2}} - (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) y_i, \\ \ddot{z}_i = \sum_{j=1, j \neq i}^n \frac{m_j [z_j - (x_i x_j + y_i y_j + z_i z_j) z_i]}{[1 - (x_i x_j + y_i y_j + z_i z_j)^2]^{3/2}} - (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) z_i, \\ x_i^2 + y_i^2 + z_i^2 = 1, \quad x_i \dot{x}_i + y_i \dot{y}_i + z_i \dot{z}_i = 0, \quad i = 1, \dots, n. \end{cases}$$

The Hamiltonian form of the equations of motion is

$$(20) \quad \begin{cases} \dot{\mathbf{q}}_i = m_i^{-1} \mathbf{p}_i, \\ \dot{\mathbf{p}}_i = \sum_{j=1, j \neq i}^n \frac{m_i m_j [\mathbf{q}_j - (\mathbf{q}_i \cdot \mathbf{q}_j) \mathbf{q}_i]}{[1 - (\mathbf{q}_i \cdot \mathbf{q}_j)^2]^{3/2}} - m_i^{-1} (\mathbf{p}_i \cdot \mathbf{p}_i) \mathbf{q}_i, \\ \mathbf{q}_i \cdot \mathbf{q}_i = 1, \quad \mathbf{q}_i \cdot \mathbf{p}_i = 0, \quad \kappa \neq 0, \quad i = 1, \dots, n. \end{cases}$$

Consequently the integral of energy has the form

$$(21) \quad \sum_{i=1}^n m_i^{-1} (\mathbf{p}_i \cdot \mathbf{p}_i) - \sum_{i=1}^n \sum_{j=1, j \neq i}^n \frac{m_i m_j \frac{\mathbf{q}_i \cdot \mathbf{q}_j}{\sqrt{\mathbf{q}_i \cdot \mathbf{q}_i} \sqrt{\mathbf{q}_j \cdot \mathbf{q}_j}}}{\sqrt{1 - \left(\frac{\mathbf{q}_i \cdot \mathbf{q}_j}{\sqrt{\mathbf{q}_i \cdot \mathbf{q}_i} \sqrt{\mathbf{q}_j \cdot \mathbf{q}_j}} \right)^2}} = 2h,$$

which, via $\mathbf{q}_i \cdot \mathbf{q}_i = 1$, $i = 1, \dots, n$, becomes

$$(22) \quad \sum_{i=1}^n m_i^{-1} (\mathbf{p}_i \cdot \mathbf{p}_i) - \sum_{i=1}^n \sum_{j=1, j \neq i}^n \frac{m_i m_j \mathbf{q}_i \cdot \mathbf{q}_j}{\sqrt{1 - (\mathbf{q}_i \cdot \mathbf{q}_j)^2}} = 2h,$$

and the integrals of the angular momentum take the form

$$(23) \quad \sum_{i=1}^n \mathbf{q}_i \times \mathbf{p}_i = \mathbf{c}.$$

Notice that sometimes we can use the simpler form (22) of the energy integral, but whenever we need to exploit the homogeneity of the potential or have to differentiate it, we must use the more complicated form (21).

3.10. Equations of motion in \mathbf{H}^2 . In this case, the force function (1) takes the form

$$(24) \quad U_{-1}(\mathbf{q}) = -\frac{1}{2} \sum_{i=1}^n \sum_{j=1, j \neq i}^n \frac{m_i m_j \frac{\mathbf{q}_i \square \mathbf{q}_j}{\sqrt{-\mathbf{q}_i \square \mathbf{q}_i} \sqrt{-\mathbf{q}_j \square \mathbf{q}_j}}}{\sqrt{\left(\frac{\mathbf{q}_i \square \mathbf{q}_j}{\sqrt{-\mathbf{q}_i \square \mathbf{q}_i} \sqrt{-\mathbf{q}_j \square \mathbf{q}_j}} \right)^2 - 1}},$$

so the equations of motion and their constraints become

$$(25) \quad m_i \ddot{\mathbf{q}}_i = \overline{\nabla}_{\mathbf{q}_i} U_{-1}(\mathbf{q}) + m_i (\dot{\mathbf{q}}_i \square \dot{\mathbf{q}}_i) \mathbf{q}_i, \quad \mathbf{q}_i \square \mathbf{q}_i = -1, \quad \mathbf{q}_i \square \dot{\mathbf{q}}_i = 0, \\ i = 1, \dots, n.$$

In terms of coordinates, the equations of motion and their constraints can be written as

$$(26) \quad \begin{cases} m_i \ddot{x}_i = \frac{\partial U_{-1}}{\partial x_i} + m_i(\dot{x}_i^2 + \dot{y}_i^2 - \dot{z}_i^2)x_i, \\ m_i \ddot{y}_i = \frac{\partial U_{-1}}{\partial y_i} + m_i(\dot{x}_i^2 + \dot{y}_i^2 - \dot{z}_i^2)y_i, \\ m_i \ddot{z}_i = -\frac{\partial U_{-1}}{\partial z_i} + m_i(\dot{x}_i^2 + \dot{y}_i^2 - \dot{z}_i^2)z_i, \\ x_i^2 + y_i^2 - z_i^2 = -1, \quad x_i \dot{x}_i + y_i \dot{y}_i - z_i \dot{z}_i = 0, \quad i = 1, \dots, n, \end{cases}$$

and by computing the gradients they become

$$(27) \quad \begin{cases} \ddot{x}_i = \sum_{j=1, j \neq i}^n \frac{m_j \frac{x_j + \frac{x_i x_j + y_i y_j - z_i z_j}{-x_i^2 - y_i^2 + z_i^2} x_i}{\sqrt{-x_i^2 - y_i^2 + z_i^2} \sqrt{-x_j^2 - y_j^2 + z_j^2}} + (\dot{x}_i^2 + \dot{y}_i^2 - \dot{z}_i^2)x_i, \\ \ddot{y}_i = \sum_{j=1, j \neq i}^n \frac{m_j \frac{y_j + \frac{x_i x_j + y_i y_j - z_i z_j}{-x_i^2 - y_i^2 + z_i^2} y_i}{\sqrt{-x_i^2 - y_i^2 + z_i^2} \sqrt{-x_j^2 - y_j^2 + z_j^2}} + (\dot{x}_i^2 + \dot{y}_i^2 - \dot{z}_i^2)y_i, \\ \ddot{z}_i = \sum_{j=1, j \neq i}^n \frac{m_j \frac{z_j + \frac{x_i x_j + y_i y_j - z_i z_j}{-x_i^2 - y_i^2 + z_i^2} z_i}{\sqrt{-x_i^2 - y_i^2 + z_i^2} \sqrt{-x_j^2 - y_j^2 + z_j^2}} + (\dot{x}_i^2 + \dot{y}_i^2 - \dot{z}_i^2)z_i, \\ x_i^2 + y_i^2 - z_i^2 = -1, \quad x_i \dot{x}_i + y_i \dot{y}_i - z_i \dot{z}_i = 0, \quad i = 1, \dots, n. \end{cases}$$

For the same reasons described in the previous subsection, we can use the constraints to write from now on the above system as

$$(28) \quad \begin{cases} \ddot{x}_i = \sum_{j=1, j \neq i}^n \frac{m_j [x_j + (x_i x_j + y_i y_j - z_i z_j)x_i]}{[(x_i x_j + y_i y_j - z_i z_j)^2 - 1]^{3/2}} + (\dot{x}_i^2 + \dot{y}_i^2 - \dot{z}_i^2)x_i, \\ \ddot{y}_i = \sum_{j=1, j \neq i}^n \frac{m_j [y_j + (x_i x_j + y_i y_j - z_i z_j)y_i]}{[(x_i x_j + y_i y_j - z_i z_j)^2 - 1]^{3/2}} + (\dot{x}_i^2 + \dot{y}_i^2 - \dot{z}_i^2)y_i, \\ \ddot{z}_i = \sum_{j=1, j \neq i}^n \frac{m_j [z_j + (x_i x_j + y_i y_j - z_i z_j)z_i]}{[(x_i x_j + y_i y_j - z_i z_j)^2 - 1]^{3/2}} + (\dot{x}_i^2 + \dot{y}_i^2 - \dot{z}_i^2)z_i, \\ x_i^2 + y_i^2 - z_i^2 = -1, \quad x_i \dot{x}_i + y_i \dot{y}_i - z_i \dot{z}_i = 0, \quad i = 1, \dots, n. \end{cases}$$

The Hamiltonian form of the equations of motion is

$$(29) \quad \begin{cases} \dot{\mathbf{q}}_i = m_i^{-1} \mathbf{p}_i, \\ \dot{\mathbf{p}}_i = \sum_{j=1, j \neq i}^n \frac{m_i m_j [\mathbf{q}_j + (\mathbf{q}_i \square \mathbf{q}_j) \mathbf{q}_i]}{[(\mathbf{q}_i \square \mathbf{q}_j)^2 - 1]^{3/2}} + m_i^{-1} (\mathbf{p}_i \square \mathbf{p}_i) \mathbf{q}_i, \\ \mathbf{q}_i \square \mathbf{q}_i = -1, \quad \mathbf{q}_i \square \mathbf{p}_i = 0, \quad \kappa \neq 0, \quad i = 1, \dots, n. \end{cases}$$

Consequently the integral of energy takes the form

$$(30) \quad \sum_{i=1}^n m_i^{-1}(\mathbf{p}_i \boxtimes \mathbf{p}_i) + \sum_{i=1}^n \sum_{j=1, j \neq i}^n \frac{m_i m_j \frac{\mathbf{q}_i \boxtimes \mathbf{q}_j}{\sqrt{-\mathbf{q}_i \boxtimes \mathbf{q}_i} \sqrt{-\mathbf{q}_j \boxtimes \mathbf{q}_j}}}{\sqrt{\left(\frac{\mathbf{q}_i \boxtimes \mathbf{q}_j}{\sqrt{-\mathbf{q}_i \boxtimes \mathbf{q}_i} \sqrt{-\mathbf{q}_j \boxtimes \mathbf{q}_j}} \right)^2 - 1}} = 2h,$$

which, via $\mathbf{q}_i \boxtimes \mathbf{q}_i = -1$, $i = 1, \dots, n$, becomes

$$(31) \quad \sum_{i=1}^n m_i^{-1}(\mathbf{p}_i \boxtimes \mathbf{p}_i) + \sum_{i=1}^n \sum_{j=1, j \neq i}^n \frac{m_i m_j \mathbf{q}_i \boxtimes \mathbf{q}_j}{\sqrt{(\mathbf{q}_i \boxtimes \mathbf{q}_j)^2 - 1}} = 2h,$$

and the integrals of the angular momentum can be written as

$$(32) \quad \sum_{i=1}^n \mathbf{q}_i \boxtimes \mathbf{p}_i = \mathbf{c}.$$

Notice that sometimes we can use the simpler form (31) of the energy integral, but whenever we need to exploit the homogeneity of the potential or have to differentiate it, we must use the more complicated form (30).

3.11. Equations of motion in \mathbf{S}^μ and \mathbf{H}^μ . The formalism we adopted in this paper allows a straightforward generalization of the n -body problem to \mathbf{S}^μ and \mathbf{H}^μ for any integer $\mu \geq 1$. The equations of motion in μ -dimensional spaces of constant curvature have the form (7) for vectors \mathbf{q}_i and \mathbf{q}_j of $\mathbf{R}^{\mu+1}$ constrained to the corresponding manifold. It is then easy to see from any coordinate-form of the system that \mathbf{S}^ν and \mathbf{H}^ν are invariant sets for the equations of motion in \mathbf{S}^μ and \mathbf{H}^μ , respectively, for any integer $\nu < \mu$.

Indeed, this is the case, say, for equations (19), if we take $x_i(0) = 0, \dot{x}_i(0) = 0$, $i = 1, \dots, n$. Then the equations of \ddot{x}_i are identically satisfied, and the motion takes place on the circle $y^2 + z^2 = 1$. The generalization of this idea from one component to any number ν of components in a $(\mu + 1)$ -dimensional space, with $\nu < \mu$, is straightforward. Therefore the study of the n -body problem on surfaces of constant curvature is fully justified.

The only aspect of this generalization that is not obvious from our formalism is how to extend the cross product to higher dimensions. But this extension can be done as in general relativity with the help of the exterior product. However, we will not get into higher dimensions in this paper. Our further goal is to study the 2-dimensional case.

4. SINGULARITIES

Singularities have always been a rich source of research in the theory of differential equations. The n -body problem we derived in the previous section seems to make no exception from this rule. In what follows, we will point out

the singularities that occur in our problem and prove some results related to them. The most surprising seems to be the existence of a class of solutions with some hybrid singularities, which are both collisional and non-collisional.

4.1. Singularities of the equations. The equations of motion (14) have restrictions. First, the variables are constrained to a surface of constant curvature, i.e. $(\mathbf{q}, \mathbf{p}) \in \mathbf{T}^*(\mathbf{M}_\kappa^2)^n$, where \mathbf{M}_κ^2 is the surface of curvature $\kappa \neq 0$ (in particular, $\mathbf{M}_1^2 = \mathbf{S}^2$ and $\mathbf{M}_{-1}^2 = \mathbf{H}^2$), $\mathbf{T}^*(\mathbf{M}_\kappa^2)^n$ is the cotangent bundle of \mathbf{M}_κ^2 . Second, system (14), which contains the gradient (8), is undefined in the set $\Delta := \cup_{1 \leq i < j \leq n} \Delta_{ij}$, with

$$\Delta_{ij} := \{\mathbf{q} \in (\mathbf{M}_\kappa^2)^n \mid (\kappa \mathbf{q}_i \odot \mathbf{q}_j)^2 = 1\},$$

where both the force function (1) and its gradient (8) become infinite. Thus the set Δ contains the singularities of the equations of motion.

The singularity condition, $(\kappa \mathbf{q}_i \odot \mathbf{q}_j)^2 = 1$, suggests that we consider two cases, and thus write $\Delta_{ij} = \Delta_{ij}^+ \cup \Delta_{ij}^-$, where

$$\Delta_{ij}^+ := \{\mathbf{q} \in (\mathbf{M}_\kappa^2)^n \mid \kappa \mathbf{q}_i \odot \mathbf{q}_j = 1\} \quad \text{and} \quad \Delta_{ij}^- := \{\mathbf{q} \in (\mathbf{M}_\kappa^2)^n \mid \kappa \mathbf{q}_i \odot \mathbf{q}_j = -1\}.$$

Accordingly, we define

$$\Delta^+ := \cup_{1 \leq i < j \leq n} \Delta_{ij}^+ \quad \text{and} \quad \Delta^- := \cup_{1 \leq i < j \leq n} \Delta_{ij}^-.$$

Then $\Delta = \Delta^+ \cup \Delta^-$. The elements of Δ^+ correspond to collisions for any $\kappa \neq 0$, whereas the elements of Δ^- correspond to what we will call antipodal singularities when $\kappa > 0$. The latter occur when two bodies are at the opposite ends of the same diameter of a sphere. For $\kappa < 0$, such singularities do not exist because $\kappa \mathbf{q}_i \odot \mathbf{q}_j \geq 1$.

In conclusion, the equations of motion are undefined for configurations that involve collisions on spheres or hyperboloids, as well as for configurations with antipodal bodies on spheres of any curvature $\kappa > 0$. In both cases, the gravitational forces become infinite.

In the 2-body problem, Δ^+ and Δ^- are disjoint sets. Indeed, since there are only two bodies, $\kappa \mathbf{q}_1 \cdot \mathbf{q}_2$ is either $+1$ or -1 , but cannot be both. The set $\Delta^+ \cap \Delta^-$, however, is not empty for $n \geq 3$. In the 3-body problem, for instance, the configuration in which two bodies are at collision and the third lies at the opposite end of the corresponding diameter is, what we will call from now on, a collision-antipodal singularity.

The theory of differential equations merely regards singularities as points for which the equations break down, and must therefore be avoided. But singularities exhibit sometimes a dynamical structure. In the 3-body problem in \mathbf{R} , for instance, the set of binary collisions is attractive in the sense that for any given initial velocities, there are initial positions such that if two bodies come close enough to each other but far enough from other collisions, then the collision will take place. (Things are more complicated with triple collisions.

Two of the bodies coming close to triple collisions may form a binary while the third gets expelled with high velocity away from the other two, [53].)

Something similar happens for binary collisions in the 3-body problem on a geodesic of \mathbf{S}^2 . Given some initial velocities, one can choose initial positions that put m_1 and m_2 close enough to a binary collision, and m_3 far enough from an antipodal singularity with either m_1 or m_2 , such that the binary collision takes place. This is indeed the case because the attraction between m_1 and m_2 can be made as large as desired by placing the bodies close enough to each other. Since m_3 is far enough from an antipodal position, and no comparable force can oppose the attraction between m_1 and m_2 , these bodies will collide.

Antipodal singularities lead to a new phenomenon on geodesics of \mathbf{S}^2 . Given initial velocities, no matter how close one chooses initial positions near an antipodal singularity, the corresponding solution is repelled in future time from this singularity as long as no collision force compensates for this force. So while binary collisions can be regarded as attractive if far away from binary antipodal singularities, binary antipodal singularities can be seen as repulsive if far away from collisions. But what happens when collision and antipodal singularities are close to each other? As we will see in the next subsection, the behavior of solutions in that region is sensitive to the choice of masses and initial conditions. In particular, we will prove the existence of some hybrid singular solutions in the 3-body problem, namely those that end in finite time in a collision-antipodal singularity, as well as of solutions that reach a collision-antipodal configuration but remain analytic at this point.

4.2. Solution singularities. The set Δ is related to singularities which arise from the question of existence and uniqueness of initial value problems. For initial conditions $(\mathbf{q}, \mathbf{p})(0) \in \mathbf{T}^*(\mathbf{M}_\kappa^2)^n$ with $\mathbf{q}(0) \notin \Delta$, standard results of the theory of differential equations ensure local existence and uniqueness of an analytic solution (\mathbf{q}, \mathbf{p}) defined on some interval $[0, t^+)$. Since the surfaces \mathbf{M}_κ^2 are connected, this solution can be analytically extended to an interval $[0, t^*)$, with $0 < t^+ \leq t^* \leq \infty$. If $t^* = \infty$, the solution is globally defined. But if $t^* < \infty$, the solution is called singular, and we say that it has a singularity at time t^* .

There is a close connection between singular solutions and singularities of the equations of motion. In the classical case ($\kappa = 0$), this connection was pointed out by Paul Painlevé towards the end of the 19th century. In his famous lectures given in Stockholm, [57], he showed that every singular solution (\mathbf{q}, \mathbf{p}) is such that $\mathbf{q}(t) \rightarrow \Delta$ when $t \rightarrow t^*$, for otherwise the solution would be globally defined. In the Euclidean case, $\kappa = 0$, the set Δ is formed by all configurations with collisions, so when $\mathbf{q}(t)$ tends to an element of Δ , the solution ends in a collision singularity. But it is also possible that $\mathbf{q}(t)$ tends to Δ without asymptotic phase, i.e. by oscillating among various elements without

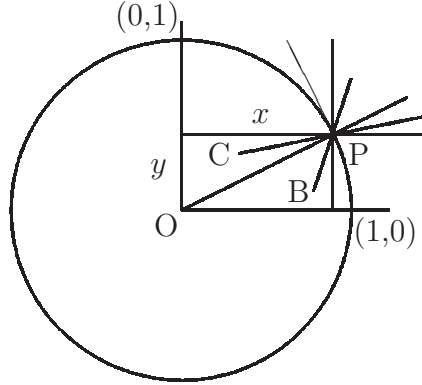


FIGURE 1. The relative positions of the force acting on m , while the body is on the geodesic $z = 0$.

ever reaching a definite position. Painlevé conjectured that such noncollision singularities, which he called pseudocollisions, exist. In 1908, Hugo von Zeipel showed that a necessary condition for a solution to experience a pseudocollision is that the motion becomes unbounded in finite time, [75], [54]. Zhihong (Jeff) Xia produced the first example of this kind in 1992, [77]. Historical accounts of this development appear in [19] and [22].

The results of Painlevé don't remain intact in our problem, [23], [21], so whether pseudocollisions exist for $\kappa \neq 0$ is not clear. Nevertheless, we will now show that there are solutions ending in collision-antipodal singularities of the equations of motion, solutions these singularities repel, as well as solutions that are not singular at such configurations. To prove these facts, we need the result stated below, which provides a criterion for determining the direction of motion along a great circle in the framework of an isosceles problem defined in an invariant set \mathbf{S}^1 .

Lemma 1. *Consider the n -body problem in \mathbf{S}^2 , and assume that a body of mass m is at rest at time t_0 on the geodesic $z = 0$ within its first quadrant, $x, y > 0$. Then, if*

(a) $\ddot{x}(t_0) > 0$ and $\ddot{y}(t_0) < 0$, *the force pulls the body along the circle toward the point $(x, y) = (1, 0)$.*

(b) $\ddot{x}(t_0) < 0$ and $\ddot{y}(t_0) > 0$, *the force pulls the body along the circle toward the point $(x, y) = (0, 1)$.*

(c) $\ddot{x}(t_0) \leq 0$ and $\ddot{y}(t_0) \leq 0$, *the force pulls the body toward the point $(1, 0)$ if $\ddot{y}(t_0)/\ddot{x}(t_0) > y(t_0)/x(t_0)$, toward $(0, 1)$ if $\ddot{y}(t_0)/\ddot{x}(t_0) < y(t_0)/x(t_0)$, but no force acts on the body if neither of the previous inequalities holds.*

(d) $\ddot{x}(t_0) > 0$ and $\ddot{y}(t_0) > 0$, *the motion is impossible.*

Proof. By equation (6), $x\ddot{x} + y\ddot{y} = -(\dot{x}^2 + \dot{y}^2) \leq 0$, which means that the force acting on m is always directed along the tangent at m to the geodesic

circle $z = 0$ or inside the half-plane containing this circle. Assuming that an xy -coordinate system is fixed at the origin of the acceleration vector (point P in Figure 1), this vector always lies in the half-plane below the line of slope $-x(t_0)/y(t_0)$ (i.e. the tangent to the circle at the point P in Figure 1). We further prove each case separately.

(a) If $\ddot{x}(t_0) > 0$ and $\ddot{y}(t_0) < 0$, the force acting on m is represented by a vector that lies in the region given by the intersection of the fourth quadrant (counted counterclockwise) and the half plane below the line of slope $-x(t_0)/y(t_0)$. Then, obviously, the force pulls the body along the circle in the direction of the point $(1, 0)$.

(b) If $\ddot{x}(t_0) < 0$ and $\ddot{y}(t_0) > 0$, the force acting on m is represented by a vector that lies in the region given by the intersection of the second quadrant and the half plane lying below the line of slope $-x(t_0)/y(t_0)$. Then, obviously, the force pulls the body along the circle in the direction of the point $(0, 1)$.

(c) If $\ddot{x}(t_0) \leq 0$ and $\ddot{y}(t_0) \leq 0$, the force acting on m is represented by a vector lying in the third quadrant. Then the direction in which this force acts depends on whether the acceleration vector lies: (i) below the line of slope $y(t_0)/x(t_0)$ (PB is below OP in Figure 1); (ii) above it (PC is above OP); or (iii) on it (i.e. on the line OP). Case (iii) includes the case when the acceleration is zero.

In case (i), the acceleration vector lies on a line whose slope is larger than $y(t_0)/x(t_0)$, i.e. $\ddot{y}(t_0)/\ddot{x}(t_0) > y(t_0)/x(t_0)$, so the force pulls m toward $(1, 0)$. In case (ii), the acceleration vector lies on a line of slope that is smaller than $y(t_0)/x(t_0)$, i.e. $\ddot{y}(t_0)/\ddot{x}(t_0) < y(t_0)/x(t_0)$, so the force pulls m toward $(0, 1)$. In case (iii), the acceleration vector is either zero or lies on the line of slope $y(t_0)/x(t_0)$, i.e. $\ddot{y}(t_0)/\ddot{x}(t_0) = y(t_0)/x(t_0)$. But the latter alternative never happens. This fact follows from the equations of motion (7), which show that the acceleration is the difference between the gradient of the force function and a multiple of the position vector. But according to Euler's formula for homogeneous functions, (3), and the fact that the velocities are zero, these vectors are orthogonal, so their difference can have the same direction as one of them only if it is zero. This vectorial argument agrees with the kinematic facts, which show that if $\dot{x}(t_0) = \dot{y}(t_0) = 0$ and the acceleration has the same direction as the position vector, then m doesn't move, so $\dot{x}(t) = \dot{y}(t) = 0$, and therefore $\ddot{x}(t) = \ddot{y}(t) = 0$ for all t . In particular, this means that when $\ddot{y}(t_0) = \ddot{x}(t_0) = 0$, no force acts on m , so the body remains fixed.

(d) If $\ddot{x}(t_0) > 0$ and $\ddot{y}(t_0) > 0$, the force acting on m is represented by a vector that lies in the region given by the intersection between the first quadrant and the half-plane lying below the line of slope $-x(t_0)/y(t_0)$. But this region is empty, so the motion doesn't take place. \square

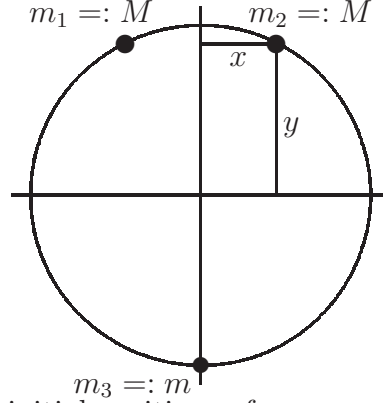


FIGURE 2. The initial positions of m_1, m_2 , and m_3 on the geodesic $z = 0$.

We will further prove the existence of solutions with collision-antipodal singularities, solutions repelled from collision-antipodal singularities in positive time, as well as of solutions that remain analytic at a collision-antipodal configuration. They show that the dynamics of $\Delta^+ \cap \Delta^-$ is more complicated than the dynamics of Δ^+ and Δ^- away from the intersection, since solutions can go both towards and away from this set for $t > 0$, and can even avoid singularities. This result represents a first example of a non-collision singularity reached by only three bodies as well as a first example of a non-singularity collision.

Theorem 1. *Consider the 3-body problem in \mathbf{S}^2 with the bodies m_1 and m_2 having mass $M > 0$ and the body m_3 having mass $m > 0$. Then*

- (i) *there are values of m and M , as well as initial conditions, for which the solutions end in finite time in a collision-antipodal singularity;*
- (ii) *other choices of masses and initial conditions lead to solutions that are repelled from a collision-antipodal singularity;*
- (iii) *and yet other choices of masses and initial data correspond to solutions that reach a collision-antipodal configuration but remain analytic at this point.*

Proof. Let us start with some initial conditions we will refine on the way. During the refinement process, we will also choose suitable masses. Consider

$$\begin{aligned} x_1(0) &= -x(0), & y_1(0) &= y(0), & z_1(0) &= 0, \\ x_2(0) &= x(0), & y_2(0) &= y(0), & z_2(0) &= 0, \\ x_3(0) &= 0, & y_3(0) &= -1, & z_3(0) &= 0, \end{aligned}$$

as well as zero initial velocities, where $0 < x(t), y(t) < 1$ are functions with $x(t)^2 + y(t)^2 = 1$. Since all z coordinates are zero, only the equations of coordinates x and y play a role in the motion. The symmetry of these initial conditions implies that m_3 remains fixed for all time (in fact the equations

corresponding to \ddot{x}_3 and \ddot{y}_3 reduce to identities), that the angular momentum is zero, and that it is enough to see what happens for m_2 , because m_1 behaves symmetrically with respect to the y axis. Thus, substituting the above initial conditions into the equations of motion, we obtain

$$(33) \quad \ddot{x}(0) = -\frac{y(0)}{x^2(0)} \left(\frac{M}{4y^2(0)} - m \right) \quad \text{and} \quad \ddot{y}(0) = \frac{1}{x(0)} \left(\frac{M}{4y^2(0)} - m \right).$$

These equations show that several situations occur, depending on the choice of masses and initial positions. Here are two significant possibilities.

1. For $M \geq 4m$, it follows that $\ddot{x}(0) < 0$ and $\ddot{y}(0) > 0$ for any choices of initial positions with $0 < x(0), y(0) < 1$.

2. For $M < 4m$, there are initial positions for which:

- (a) $\ddot{x}(0) < 0$ and $\ddot{y}(0) > 0$,
- (b) $\ddot{x}(0) > 0$ and $\ddot{y}(0) < 0$,
- (c) $\ddot{x}(0) = \ddot{y}(0) = 0$.

In case 2(c), the solutions are fixed points of the equations of motion, a situation achieved, for instance, when $M = 2m$ and $x(0) = y(0) = \sqrt{2}/2$. The cases of interest for us, however, are 1 and 2(b). In the former, m_2 begins to move from rest towards a collision with m_1 at $(0, 1)$, but whether this collision takes place also depends on velocities, which affect the equations of motion. In the latter case, m_2 moves away from the same collision, and we need to see again how the velocities alter this initial tendency. So let us write now the equations of motion for m_2 starting from arbitrary masses M and m . The computations lead us to the system

$$(34) \quad \begin{cases} \ddot{x} = -\frac{M}{4x^2y} + \frac{my}{x^2} - (\dot{x}^2 + \dot{y}^2)x \\ \ddot{y} = \frac{M}{4xy^2} - \frac{m}{x} - (\dot{x}^2 + \dot{y}^2)y \end{cases}$$

and the energy integral

$$\dot{x}^2 + \dot{y}^2 = \frac{h}{M} - \frac{2my}{x} + \frac{M(2y^2 - 1)}{2xy}.$$

Substituting this expression of $\dot{x}^2 + \dot{y}^2$ into equations (34), we obtain

$$(35) \quad \begin{cases} \ddot{x} = \frac{4(M-2m)x^4 - 2(M-2m)x^2 - M + 4m}{4x^2y} - \frac{h}{M}x \\ \ddot{y} = \frac{M + 2(M-2m)y^2 - 4(M-2m)y^4}{4xy^2} - \frac{h}{M}y. \end{cases}$$

We will further focus on the first class of orbits announced in this theorem.

(i) To prove the existence of solutions with collision-antipodal singularities, let us further examine the case $M = 8m$, which brings system (35) to the form

$$(36) \quad \begin{cases} \ddot{x} = \frac{6mx^2}{y} - \frac{3m}{y} - \frac{m}{x^2y} - \frac{h}{8m}x \\ \ddot{y} = \frac{2m}{xy^2} + \frac{3m}{x} - \frac{6my^2}{x} - \frac{h}{8m}y, \end{cases}$$

with the energy integral

$$(37) \quad \dot{x}^2 + \dot{y}^2 + \frac{4mx}{y} - \frac{2my}{x} = \frac{h}{8m}.$$

Then, as $x \rightarrow 0$ and $y \rightarrow 1$, both \ddot{x} and \ddot{y} tend to $-\infty$, so they are ultimately negative. This fact corresponds to case (c) of Lemma 1. But a simple computation shows that \ddot{y}/\ddot{x} tends to zero as $x \rightarrow 0$ and $y \rightarrow 1$. Since $y/x > 0$, it follows that if $(x(0), y(0))$ is chosen close enough to $(0, 1)$, then $\ddot{y}(0)/\ddot{x}(0) < y(0)/x(0)$, so according to the conclusion of Lemma 1(c) the collision-antipodal configuration is reached. As the forces and the potential are infinite at this point, using the energy relation (37) it follows that the velocities are also infinite. Consequently the motion cannot be analytically extended beyond the collision-antipodal configuration, which thus proves to be a singularity.

(ii) To show the existence of solutions repelled from a collision-antipodal singularity of the equations of motion in positive time, let us take $M = 2m$. Then equations (35) have the form

$$(38) \quad \begin{cases} \ddot{x} = \frac{m}{2x^2y} - \frac{h}{2m}x \\ \ddot{y} = \frac{m}{2xy^2} - \frac{h}{2m}y, \end{cases}$$

with the integral of energy

$$(39) \quad \dot{x}^2 + \dot{y}^2 + \frac{m}{xy} = \frac{h}{2m},$$

which implies that $h > 0$. Obviously, as $x \rightarrow 0$ and $y \rightarrow 1$, the forces and the kinetic energy become infinite, so the collision-antipodal configuration is a singularity if it were to be reached. But as we will further see, this cannot happen for this choice of masses. Indeed, as we saw in case 2(c) above, the initial position $x(0) = y(0) = \sqrt{2}/2$ corresponds to a fixed point of the equations of motion for zero initial velocities. Therefore we must seek the desired solution for initial conditions with $0 < x(0) < \sqrt{2}/2$ and the corresponding choice of $y(0) > 0$. Let us pick any such initial positions, as close to the collision-antipodal singularity as we want, and zero initial velocities. For $x \rightarrow 0$, however, equations (38) show that both \ddot{x} and \ddot{y} grow positive. But according to case (d) of Lemma 1, such an outcome is impossible, so the motion cannot come infinitesimally close to the corresponding collision-antipodal singularity, which repels any solution with $M = 2m$ and initial conditions chosen as we previously described.

(iii) To prove the existence of solutions that have no singularity at a collision-antipodal configuration, let us further examine the case $M = 4m$, which brings

system (35) to the form

$$(40) \quad \begin{cases} \ddot{x} = \frac{m(2x^2-1)}{y} - \frac{h}{4m}x \\ \ddot{y} = \frac{mx(2y^2+1)}{y^2} - \frac{h}{4m}y. \end{cases}$$

For this choice of masses, the energy integral becomes

$$(41) \quad \dot{x}^2 + \dot{y}^2 + \frac{2mx}{y} = \frac{h}{4m}.$$

We can compute the value of h from the initial conditions. Thus, for initial positions $x(0), y(0)$ and initial velocities $\dot{x}(0) = \dot{y}(0) = 0$, the energy constant is $h = 8m^2x(0)/y(0) > 0$.

Assuming that $x \rightarrow 0$ and $y \rightarrow 1$, equations (40) imply that $\ddot{x}(t) \rightarrow -m < 0$ and $\ddot{y}(t) \rightarrow -h/4m < 0$, which means that the forces are finite at the collision-antipodal configuration. We are thus in the case (c) of Lemma 1, so to determine the direction of motion for m_2 when it comes close to $(0, 1)$, we need to take into account the ratio \ddot{y}/\ddot{x} , which tends to $h/4m^2$ as $x \rightarrow 0$. Since $h = 8m^2x(0)/y(0)$, $\lim_{x \rightarrow 0}(\ddot{y}/\ddot{x}) = 2x(0)/y(0)$. Then $2x(0)/y(0) < y(0)/x(0)$ for any $x(0)$ and $y(0)$ with $0 < x(0) < 1/\sqrt{3}$ and the corresponding choice of $y(0) > 0$ given by the constraint $x^2(0) + y^2(0) = 1$. But the inequality $2x(0)/y(0) < y(0)/x(0)$ is equivalent to the condition $\ddot{y}(t_0)/\ddot{x}(t_0) < y(t_0)/x(t_0)$ in Lemma 1(c), according to which the force pulls m_2 toward $(0, 1)$. Therefore the velocity and the force acting on m_2 keep this body on the same path until the collision-antipodal configuration occurs.

It is also clear from equation (41) that the velocity is positive and finite at collision. Since the distance between the initial position and $(0, 1)$ is bounded, m_2 collides with m_1 in finite time. Therefore the choice of masses with $M = 4m$, initial positions $x(0), y(0)$ with $0 < x(0) < 1/\sqrt{3}$ and the corresponding value of $y(0)$, and initial velocities $\dot{x}(0) = \dot{y}(0) = 0$, leads to a solution that remains analytic at the collision-antipodal configuration, so the motion naturally extends beyond this point. \square

5. RELATIVE EQUILIBRIA IN \mathbf{S}^2

In this section we will prove a few results related to fixed points and elliptic relative equilibria in \mathbf{S}^2 . Since, by Euler's theorem (see Appendix), every element of the group $SO(3)$ can be written, in an orthonormal basis, as a rotation about the z axis, we can define elliptic relative equilibria as follows.

Definition 1. *An elliptic relative equilibrium in \mathbf{S}^2 is a solution of the form $\mathbf{q}_i = (x_i, y_i, z_i)$, $i = 1, \dots, n$, of equations (19) with $x_i = r_i \cos(\omega t + \alpha_i)$, $y_i = r_i \sin(\omega t + \alpha_i)$, $z_i = \text{constant}$, where ω, α_i , and r_i , with $0 \leq r_i = (1 - z_i^2)^{1/2} \leq 1$, $i = 1, \dots, n$, are constants.*

Notice that although the equations of motion don't have an integral of the center of mass, a "weak" property of this kind occurs for elliptic relative equilibria. Indeed, it is easy to see that if all the bodies are at all times on one side of a plane containing the rotation axis, then the integrals of the angular momentum are violated. This happens because under such circumstances the vector representing the total angular momentum cannot be zero or parallel to the z axis.

5.1. Fixed points. The simplest solutions of the equations of motion are fixed points. They can be seen as trivial relative equilibria that correspond to $\omega = 0$. In terms of the equations of motion, we can define them as follows.

Definition 2. *A solution of system (20) is called a fixed point if*

$$\nabla_{\mathbf{q}_i} U_1(\mathbf{q})(t) = \mathbf{p}_i(t) = \mathbf{0} \quad \text{for all } t \in \mathbf{R} \quad \text{and } i = 1, \dots, n.$$

Let us start with finding the simplest fixed points, namely those that occur when all the masses are equal.

Theorem 2. *Consider the n -body problem in \mathbf{S}^2 with n odd. If the masses are all equal, the regular n -gon lying on any geodesic is a fixed point of the equations of motion. For $n = 4$, the regular tetrahedron is a fixed point too.*

Proof. Assume that $m_1 = m_2 = \dots = m_n$, and consider an n -gon with an odd number of sides inscribed in a geodesic of \mathbf{S}^2 with a body, initially at rest, at each vertex. In general, two forces act on the body of mass m_i : the force $\nabla_{\mathbf{q}_i} U_1(\mathbf{q})$, which is due to the interaction with the other bodies, and the force $-m_i(\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i)\mathbf{q}_i$, which is due to the constraints. The latter force is zero at $t = 0$ because the bodies are initially at rest. Since $\mathbf{q}_i \cdot \nabla_{\mathbf{q}_i} U_1(\mathbf{q}) = 0$, it follows that $\nabla_{\mathbf{q}_i} U_1(\mathbf{q})$ is orthogonal to \mathbf{q}_i , and thus tangent to \mathbf{S}^2 . Then the symmetry of the n -gon implies that, at the initial moment $t = 0$, $\nabla_{\mathbf{q}_i} U_1(\mathbf{q})$ is the sum of pairs of forces, each pair consisting of opposite forces that cancel each other. This means that $\nabla_{\mathbf{q}_i} U_1(\mathbf{q}) = \mathbf{0}$. Therefore, from the equations of motion and the fact that the bodies are initially at rest, it follows that

$$\ddot{\mathbf{q}}_i(0) = -(\dot{\mathbf{q}}_i(0) \cdot \dot{\mathbf{q}}_i(0))\mathbf{q}_i(0) = \mathbf{0}, \quad i = 1, \dots, n.$$

But then no force acts on the body of mass m_i at time $t = 0$, consequently the body doesn't move. So $\dot{\mathbf{q}}_i(t) = \mathbf{0}$ for all $t \in \mathbf{R}$. Then $\ddot{\mathbf{q}}_i(t) = \mathbf{0}$ for all $t \in \mathbf{R}$, therefore $\nabla_{\mathbf{q}_i} U_1(\mathbf{q})(t) = \mathbf{0}$ for all $t \in \mathbf{R}$, so the n -gon is a fixed point of equations (19).

Notice that if n is even, the n -gon has $n/2$ pairs of antipodal vertices. Since antipodal bodies introduce singularities into the equations of motion, only the n -gons with an odd number of vertices are fixed points of equations (19).

The proof that the regular tetrahedron is a fixed point can be merely done by computing that 4 bodies of equal masses with initial coordinates given

by $\mathbf{q}_1 = (0, 0, 1)$, $\mathbf{q}_2 = (0, 2\sqrt{2}/3, -1/3)$, $\mathbf{q}_3 = (-2/\sqrt{6}, -\sqrt{2}/3, -1/3)$, $\mathbf{q}_4 = (2/\sqrt{6}, -\sqrt{2}/3, -1/3)$, satisfy system (19), or by noticing that the forces acting on each body cancel each other because of the involved symmetry. \square

Remark 1. If equal masses are placed at the vertices of the other four regular polyhedra: octahedron (6 bodies), cube (8 bodies), dodecahedron (12 bodies), and icosahedron (20 bodies), they do not form fixed points because antipodal singularities occur in each case.

Remark 2. In the proof of Theorem 1, we discovered that if one body has mass m and the other two mass $M = 2m$, then the isosceles triangle with the vertices at $(0, -1, 0)$, $(-\sqrt{2}/2, \sqrt{2}/2, 0)$, and $(\sqrt{2}/2, \sqrt{2}/2, 0)$ is a fixed point. Therefore one might expect that fixed points can be found for any given masses. But, as formula (33) shows, this is not the case. Indeed, if one body has mass m and the other two have masses $M \geq 4m$, there is no configuration (which must be isosceles due to symmetry) that corresponds to a fixed point since \ddot{x} and \ddot{y} are never zero. This observation proves that in the 3-body problem, there are choices of masses for which the equations of motion lack fixed points.

The following statement is an obvious consequence of the proof given for Theorem 2.

Corollary 1. *Consider an odd number of equal bodies, initially at the vertices of a regular n -gon inscribed in a great circle of \mathbf{S}^2 , and assume that the solution generated from this initial position maintains the same relative configuration for all times. Then, for all $t \in \mathbf{R}$, this solution satisfies the conditions $\nabla_{\mathbf{q}_i} U_1(\mathbf{q}(t)) = \mathbf{0}$, $i = 1, \dots, n$.*

It is interesting to see that if the bodies are within a hemisphere (meaning half a sphere and its geodesic boundary), fixed points do not occur if at least one body is not on the boundary. Let us formally state and prove this result.

Theorem 3. *Consider an initial nonsingular configuration of the n -body problem in \mathbf{S}^2 for which all bodies lie within a hemisphere, meant to include its geodesic boundary, with at least one body not on this geodesic. Then this configuration is not a fixed point.*

Proof. Without loss of generality we can consider the initial configuration of the bodies m_1, \dots, m_n in the hemisphere $z \geq 0$, whose boundary is the geodesic $z = 0$. Then at least one body has the smallest z coordinate, and let m_1 be one of these bodies. Also, at least one body has its z coordinate positive, and let m_2 be one of them. Since all initial velocities are zero, only the mutual forces between bodies act on m_1 . Then, according to the equations of motion (17), $m_1 \ddot{z}_1(0) = \frac{\partial}{\partial z_1} U_1(\mathbf{q}(0))$. But as no body has its z coordinate smaller than z_1 ,

the terms contained in the expression of $\frac{\partial}{\partial z_1} U_1(\mathbf{q}(0))$ that involve interactions between m_1 and m_i are all larger than or equal to zero for $i = 3, 4, \dots, n$, while the term involving m_2 is strictly positive. Therefore $\frac{\partial}{\partial z_1} U_1(\mathbf{q}(0)) > 0$, so m_1 moves upward the hemisphere. Consequently the initial configuration is not a fixed point. \square

5.2. Polygonal solutions. We will further show that fixed points lying on geodesics of spheres can generate relative equilibria.

Theorem 4. *Consider a fixed point given by the masses m_1, m_2, \dots, m_n that lie on a great circle of \mathbf{S}^2 . Then for every nonzero angular velocity, this configuration generates a relative equilibrium along the great circle.*

Proof. Without loss of generality, we assume that the great circle is the equator $z = 0$ and that for some given masses m_1, m_2, \dots, m_n there exist $\alpha_1, \alpha_2, \dots, \alpha_n$ such that the configuration $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_n)$ given by $\mathbf{q}_i = (x_i, y_i, 0)$, $i = 1, \dots, n$, with

$$(42) \quad x_i = \cos(\omega t + \alpha_i), y_i = \sin(\omega t + \alpha_i), \quad i = 1, \dots, n,$$

is a fixed point for $\omega = 0$. This configuration can also be interpreted as being $\mathbf{q}(0)$, i.e. the solution \mathbf{q} at $t = 0$ for any $\omega \neq 0$. So we can conclude that $\nabla_{\mathbf{q}_i} U_1(\mathbf{q}(0)) = \mathbf{0}$, $i = 1, \dots, n$. But then, for $t = 0$, the equations of motion (17) reduce to

$$(43) \quad \begin{cases} \ddot{x}_i = -(\dot{x}_i^2 + \dot{y}_i^2)x_i \\ \ddot{y}_i = -(\dot{x}_i^2 + \dot{y}_i^2)y_i, \end{cases}$$

$i = 1, \dots, n$. Notice, however, that $\dot{x}_i = -\omega \sin(\omega t + \alpha_i)$, $\ddot{x}_i = -\omega^2 \cos(\omega t + \alpha_i)$, $\dot{y}_i = \omega \cos(\omega t + \alpha_i)$, and $\ddot{y}_i = -\omega^2 \sin(\omega t + \alpha_i)$, therefore $\dot{x}_i^2 + \dot{y}_i^2 = \omega^2$. Using these computations, it is easy to see that \mathbf{q} given by (42) is a solution of (43) for every t , so no forces due to the constraints act on the bodies, neither at $t = 0$ nor later. Since $\nabla_{\mathbf{q}_i} U_1(\mathbf{q}(0)) = \mathbf{0}$, $i = 1, \dots, n$, it follows that the gravitational forces are in equilibrium at the initial moment, so no gravitational forces act on the bodies either. Consequently, the rotation imposed by $\omega \neq 0$ makes the system move like a rigid body, so the gravitational forces further remain in equilibrium, consequently $\nabla_{\mathbf{q}_i} U_1(\mathbf{q}(t)) = \mathbf{0}$, $i = 1, \dots, n$, for all t . Therefore \mathbf{q} given by (42) satisfies equations (17). Then, by Definition 1, \mathbf{q} is an elliptic relative equilibrium. \square

The following result shows that relative equilibria generated by fixed points obtained from regular n -gons on a great circle of \mathbf{S}^2 can occur only when the bodies rotate along the great circle.

Theorem 5. *Consider an odd number of equal bodies, initially at the vertices of a regular n -gon inscribed in a great circle of \mathbf{S}^2 . Then the only elliptic*

relative equilibria that can be generated from this configuration are the ones that rotate in the plane of the original great circle.

Proof. Without loss of generality, we can prove this result for the equator $z = 0$. Consider therefore an elliptic relative equilibrium solution of the form

$$(44) \quad x_i = r_i \cos(\omega t + \alpha_i), \quad y_i = r_i \sin(\omega t + \alpha_i), \quad z_i = \pm(1 - r_i^2)^{1/2},$$

$i = 1, \dots, n$, with $+$ taken for $z_i > 0$ and $-$ for $z_i < 0$. The only condition we impose on this solution is that r_i and α_i , $i = 1, \dots, n$, are chosen such that the configuration is a regular n -gon inscribed in a moving great circle of \mathbf{S}^2 at all times. Therefore the plane of the n -gon can have any angle with, say, the z -axis. This solution has the derivatives

$$\begin{aligned} \dot{x}_i &= -r_i \omega \sin(\omega t + \alpha_i), \quad \dot{y}_i = r_i \omega \cos(\omega t + \alpha_i), \quad \dot{z}_i = 0, \quad i = 1, \dots, n, \\ \ddot{x}_i &= -r_i \omega^2 \cos(\omega t + \alpha_i), \quad \ddot{y}_i = -r_i \omega^2 \sin(\omega t + \alpha_i), \quad \ddot{z}_i = 0, \quad i = 1, \dots, n. \end{aligned}$$

Then

$$\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2 = r_i^2 \omega^2, \quad i = 1, \dots, n.$$

Since, by Corollary 1, any n -gon solution with n odd satisfies the conditions

$$\nabla_{\mathbf{q}_i} U_1(\mathbf{q}) = \mathbf{0}, \quad i = 1, \dots, n,$$

system (19) reduces to

$$\begin{cases} \ddot{x}_i = -(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)x_i, \\ \ddot{y}_i = -(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)y_i, \\ \ddot{z}_i = -(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)z_i, \quad i = 1, \dots, n. \end{cases}$$

Then the substitution of (44) into the above equations leads to:

$$\begin{cases} r_i(1 - r_i^2)\omega^2 \cos(\omega t + \alpha_i) = 0, \\ r_i(1 - r_i^2)\omega^2 \sin(\omega t + \alpha_i) = 0, \quad i = 1, \dots, n. \end{cases}$$

But assuming $\omega \neq 0$, this system is nontrivially satisfied if and only if $r_i = 1$, conditions which are equivalent to $z_i = 0$, $i = 1, \dots, n$. Therefore the bodies must rotate along the equator $z = 0$. \square

Theorem 5 raises the question whether elliptic relative equilibria given by regular polygons can rotate on other curves than geodesics. The answer is given by the following result.

Theorem 6. *Consider the n -body problem with equal masses in \mathbf{S}^2 . Then, for any n odd, $m > 0$ and $z \in (-1, 1)$, there are a positive and a negative ω that produce elliptic relative equilibria in which the bodies are at the vertices of an n -gon rotating in the plane $z = \text{constant}$. If n is even, this property is still true if we exclude the case $z = 0$.*

Proof. There are two cases to discuss: (i) n odd and (ii) n even.

(i) To simplify the presentation, we further denote the bodies by $m_i, i = -s, -s+1, \dots, -1, 0, 1, \dots, s-1, s$, where s is a positive integer, and assume that they all have mass m . Without loss of generality we can further substitute into equations (19) a solution of the form (44) with i as above, $\alpha_{-s} = -\frac{4s\pi}{2s+1}, \dots, \alpha_{-1} = -\frac{2\pi}{2s+1}, \alpha_0 = 0, \alpha_1 = \frac{2\pi}{2s+1}, \dots, \alpha_s = \frac{4s\pi}{2s+1}$, $r := r_i, z := z_i$, and consider only the equations for $i = 0$. The study of this case suffices due to the involved symmetry, which yields the same conclusions for any value of i .

The equation corresponding to the z_0 coordinate takes the form

$$\sum_{j=-s, j \neq 0}^s \frac{m(z - k_{0j}z)}{(1 - k_{0j}^2)^{3/2}} - r^2 \omega^2 z = 0,$$

where $k_{0j} = x_0 x_j + y_0 y_j + z_0 z_j = \cos \alpha_j - z^2 \cos \alpha_j + z^2$. Using the fact that $r^2 + z^2 = 1$, $\cos \alpha_j = \cos \alpha_{-j}$, and $k_{0j} = k_{0(-j)}$, this equation becomes

$$(45) \quad \sum_{j=1}^s \frac{2(1 - \cos \alpha_j)}{(1 - k_{0j}^2)^{3/2}} = \frac{\omega^2}{m}.$$

Now we need to check whether the equations corresponding to x_0 and y_0 lead to the same equation. In fact, checking for x_0 , and ignoring y_0 , suffices due to the same symmetry reasons invoked earlier or the duality of the trigonometric functions \sin and \cos . The substitution of the the above functions into the first equation of (19) leads us to

$$(r^2 - 1)\omega^2 \cos \omega t = \sum_{j=-s, j \neq 0}^s \frac{m[\cos(\omega t + \alpha_j) - k_{0j} \cos \omega t]}{(1 - k_{0j}^2)^{3/2}}.$$

A straightforward computation, which uses the fact that $r^2 + z^2 = 1$, $\sin \alpha_j = -\sin \alpha_{-j}$, $\cos \alpha_j = \cos \alpha_{-j}$, and $k_{0j} = k_{0(-j)}$, yields the same equation (45). Writing the denominator of equation (45) explicitly, we are led to

$$(46) \quad \sum_{j=1}^s \frac{2}{(1 - \cos \alpha_j)^{1/2} (1 - z^2)^{3/2} [2 - (1 - \cos \alpha_j)(1 - z^2)]^{3/2}} = \frac{\omega^2}{m}.$$

The left hand side is always positive, so for any $m > 0$ and $z \in (-1, 1)$ fixed, there are a positive and a negative ω that satisfy the equation. Therefore the n -gon with an odd number of sides is an elliptic relative equilibrium.

(ii) To simplify the presentation when n is even, we denote the bodies by $m_i, i = -s+1, \dots, -1, 0, 1, \dots, s-1, s$, where s is a positive integer, and assume that they all have mass m . Without loss of generality, we can substitute into equations (19) a solution of the form (44) with i as above, $\alpha_{-s+1} = \frac{(-s+1)\pi}{s}, \dots, \alpha_{-1} = -\frac{\pi}{s}, \alpha_0 = 0, \alpha_1 = \frac{\pi}{s}, \dots, \alpha_{s-1} = \frac{(s-1)\pi}{s}, \alpha_s = \pi$,

$r := r_i$, $z := z_i$, and consider as in the previous case only the equations for $i = 0$. Then using the fact that $k_{0j} = k_{0(-j)}$, $\cos \alpha_j = \cos \alpha_{-j}$, and $\cos \pi = -1$, a straightforward computation brings the equation corresponding to z_0 to the form

$$(47) \quad \sum_{j=1}^{s-1} \frac{2(1 - \cos \alpha_j)}{(1 - k_{0j}^2)^{3/2}} + \frac{2}{(1 - k_{0s}^2)^{3/2}} = \frac{\omega^2}{m}.$$

Using additionally the relations $\sin \alpha_j = -\sin \alpha_{-j}$ and $\sin \pi = 0$, we obtain for the equation corresponding to x_0 the same form (47), which—for k_{0j} and k_{0s} written explicitly—becomes

$$\sum_{j=1}^{s-1} \frac{2}{(1 - \cos \alpha_j)^{1/2} (1 - z^2)^{3/2} [2 - (1 - \cos \alpha_j)(1 - z^2)]^{3/2}} + \frac{1}{4z^2 |z| (1 - z^2)^{3/2}} = \frac{\omega^2}{m}.$$

Since the left hand side of this equations is positive and finite, given any $m > 0$ and $z \in (-1, 0) \cup (0, 1)$, there are a positive and a negative ω that satisfy it. So except for the case $z = 0$, which introduces antipodal singularities, the rotating n -gon with an even number of sides is an elliptic relative equilibrium. \square

5.3. Lagrangian solutions. The case $n = 3$ presents particular interest in the Euclidean case because the equilateral triangle is an elliptic relative equilibrium for any values of the masses, not only when the masses are equal. But before we check whether this fact holds in \mathbf{S}^2 , let us consider the case of three equal masses in more detail.

Corollary 2. *Consider the 3-body problem with equal masses, $m := m_1 = m_2 = m_3$, in \mathbf{S}^2 . Then for any $m > 0$ and $z \in (-1, 1)$, there are a positive and a negative ω that produce elliptic relative equilibria in which the bodies are at the vertices of an equilateral triangle that rotates in the plane $z = \text{constant}$. Moreover, for every ω^2/m there are two values of z that lead to relative equilibria if $\omega^2/m \in (8/\sqrt{3}, \infty) \cup \{3\}$, three values if $\omega^2/m = 8/\sqrt{3}$, and four values if $\omega^2/m \in (3, 8/\sqrt{3})$.*

Proof. The first part of the statement is a consequence of Theorem 6 for $n = 3$. Alternatively, we can substitute into system (19) a solution of the form (44) with $i = 1, 2, 3$, $r := r_1 = r_2 = r_3$, $z = \pm(1 - r^2)^{1/2}$, $\alpha_1 = 0, \alpha_2 = 2\pi/3, \alpha_3 = 4\pi/3$, and obtain the equation

$$(48) \quad \frac{8}{\sqrt{3}(1 + 2z^2 - 3z^4)^{3/2}} = \frac{\omega^2}{m}.$$

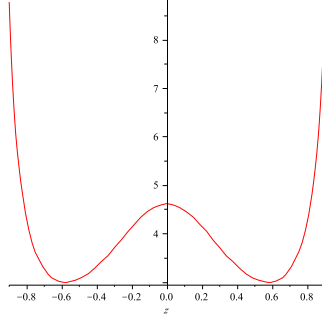


FIGURE 3. The graph of the function $f(z) = \frac{8}{\sqrt{3}(1+2z^2-3z^4)^{3/2}}$ for $z \in (-1, 1)$.

The left hand side is positive for $z \in (-1, 1)$ and tends to infinity when $z \rightarrow \pm 1$ (see Figure 3). So for any z in this interval and $m > 0$, there are a positive and a negative ω for which the above equation is satisfied. Figure 3 and a straightforward computation also clarify the second part of the statement. \square

Remark 3. A result similar to Corollary 2 can be proved for two equal masses that rotate on a non-geodesic circle, when the bodies are situated at opposite ends of a rotating diameter. Then, for $z \in (-1, 0) \cup (0, 1)$, the analogue of (48) is the equation

$$\frac{1}{4z^2|z|(1-z^2)^{3/2}} = \frac{\omega^2}{m}.$$

The case $z = 0$ yields no solution because it involves an antipodal singularity.

We have reached now the point when we can decide whether the equilateral triangle can be an elliptic relative equilibrium in \mathbf{S}^2 if the masses are not equal. The following result shows that, unlike in the Euclidean case, the answer is negative when the bodies move on the sphere in the same Euclidean plane.

Proposition 2. *In the 3-body problem in \mathbf{S}^2 , if the bodies m_1, m_2, m_3 are initially at the vertices of an equilateral triangle in the plane $z = \text{constant}$ for some $z \in (-1, 1)$, then there are initial velocities that lead to an elliptic relative equilibrium in which the triangle rotates in its own plane if and only if $m_1 = m_2 = m_3$.*

Proof. The implication which shows that if $m_1 = m_2 = m_3$, the rotating equilateral triangle is a relative equilibrium, follows from Theorem 2. To prove the other implication, we substitute into equations (19) a solution of the form (44) with $i = 1, 2, 3$, $r := r_1, r_2, r_3$, $z := z_1 = z_2 = z_3 = \pm(1 - r^2)^{1/2}$,

and $\alpha_1 = 0, \alpha_2 = 2\pi/3, \alpha_3 = 4\pi/3$. The computations then lead to the system

$$(49) \quad \begin{cases} m_1 + m_2 = \gamma\omega^2 \\ m_2 + m_3 = \gamma\omega^2 \\ m_3 + m_1 = \gamma\omega^2, \end{cases}$$

where $\gamma = \sqrt{3}(1 + 2z^2 - 3z^4)^{3/2}/4$. But for any $z = \text{constant}$ in the interval $(-1, 1)$, the above system has a solution only for $m_1 = m_2 = m_3 = \gamma\omega^2/2$. Therefore the masses must be equal. \square

The next result leads to the conclusion that Lagrangian solutions in \mathbf{S}^2 can take place only in Euclidean planes of \mathbf{R}^3 . This property is known to be true in the Euclidean case for all elliptic relative equilibria, [76], but Wintner's proof doesn't work in our case because it uses the integral of the center of mass. Most importantly, our result also implies that Lagrangian orbits with non-equal masses cannot exist in \mathbf{S}^2 .

Theorem 7. *For all Lagrangian solutions in \mathbf{S}^2 , the masses m_1, m_2 and m_3 have to rotate on the same circle, whose plane must be orthogonal to the rotation axis, and therefore $m_1 = m_2 = m_3$.*

Proof. Consider a Lagrangian solution in \mathbf{S}^2 with bodies of masses m_1, m_2 , and m_3 . This means that the solution, which is an elliptic relative equilibrium, must have the form

$$\begin{aligned} x_1 &= r_1 \cos \omega t, & y_1 &= r_1 \sin \omega t, & z_1 &= (1 - r_1^2)^{1/2}, \\ x_2 &= r_2 \cos(\omega t + a), & y_2 &= r_2 \sin(\omega t + a), & z_2 &= (1 - r_2^2)^{1/2}, \\ x_3 &= r_3 \cos(\omega t + b), & y_3 &= r_3 \sin(\omega t + b), & z_3 &= (1 - r_3^2)^{1/2}, \end{aligned}$$

with $b > a > 0$. In other words, we assume that this equilateral forms a constant angle with the rotation axis, z , such that each body describes its own circle on \mathbf{S}^2 . But for such a solution to exist, it is necessary that the total angular momentum is either zero or is given by a vector parallel with the z axis. Otherwise this vector rotates around the z axis, in violation of the angular-momentum integrals. This means that at least the first two components of the vector $\sum_{i=1}^3 m_i \mathbf{q}_i \times \dot{\mathbf{q}}_i$ must be zero. A straightforward computation shows this constraint to lead to the condition

$$m_1 r_1 z_1 \sin \omega t + m_2 r_2 z_2 \sin(\omega t + a) + m_3 r_3 z_3 \sin(\omega t + b) = 0,$$

assuming that $\omega \neq 0$. For $t = 0$, this equation becomes

$$(50) \quad m_2 r_2 z_2 \sin a = -m_3 r_3 z_3 \sin b.$$

Using now the fact that

$$\alpha := x_1 x_2 + y_1 y_2 + z_1 z_2 = x_1 x_3 + y_1 y_3 + z_1 z_3 = x_3 x_2 + y_3 y_2 + z_3 z_2$$

is constant because the triangle is equilateral, the equation of the system of motion corresponding to \ddot{y}_1 takes the form

$$Kr_1(r_1^2 - 1)\omega^2 \sin \omega t = m_2 r_2 \sin(\omega t + a) + m_3 r_3 \sin(\omega t + b),$$

where K is a nonzero constant. For $t = 0$, this equation becomes

$$(51) \quad m_2 r_2 \sin a = -m_3 r_3 \sin b.$$

Dividing (50) by (51), we obtain that $z_2 = z_3$. Similarly, we can show that $z_1 = z_2 = z_3$, therefore the motion must take place in the same Euclidian plane on a circle orthogonal to the rotation axis. Proposition 2 then implies that $m_1 = m_2 = m_3$. \square

5.4. Eulerian solutions. It is now natural to ask whether such elliptic relative equilibria exist, since—as Theorem 5 shows—they cannot be generated from regular n -gons. The answer in the case $n = 3$ of equal masses is given by the following result.

Theorem 8. *Consider the 3-body problem in \mathbf{S}^2 with equal masses, $m := m_1 = m_2 = m_3$. Fix the body of mass m_1 at $(0, 0, 1)$ and the bodies of masses m_2 and m_3 at the opposite ends of a diameter on the circle $z = \text{constant}$. Then, for any $m > 0$ and $z \in (-0.5, 0) \cup (0, 1)$, there are a positive and a negative ω that produce elliptic relative equilibria.*

Proof. Substituting into the equations of motion (19) a solution of the form

$$x_1 = 0, \quad y_1 = 0, \quad z_1 = 1,$$

$$x_2 = r \cos \omega t, \quad y_2 = r \sin \omega t, \quad z_2 = z,$$

$$x_3 = r \cos(\omega t + \pi), \quad y_3 = r \sin(\omega t + \pi), \quad z_3 = z,$$

with $r \geq 0$ and z constants satisfying $r^2 + z^2 = 1$, leads either to identities or to the algebraic equation

$$(52) \quad \frac{4z + |z|^{-1}}{4z^2(1 - z^2)^{3/2}} = \frac{\omega^2}{m}.$$

The function on the left hand side is negative for $z \in (-1, -0.5)$, 0 at $z = -0.5$, positive for $z \in (-0.5, 0) \cup (0, 1)$, and undefined at $z = 0$. Therefore, for every $m > 0$ and $z \in (-0.5, 0) \cup (0, 1)$, there are a positive and a negative ω that lead to a geodesic relative equilibrium. For $z = -0.5$, we recover the equilateral fixed point. The sign of ω determines the sense of rotation. \square

Remark 4. For every $\omega^2/m \in (0, 64\sqrt{15}/45)$, there are three values of z that satisfy relation (52): one in the interval $(-0.5, 0)$ and two in the interval $(0, 1)$ (see Figure 4).

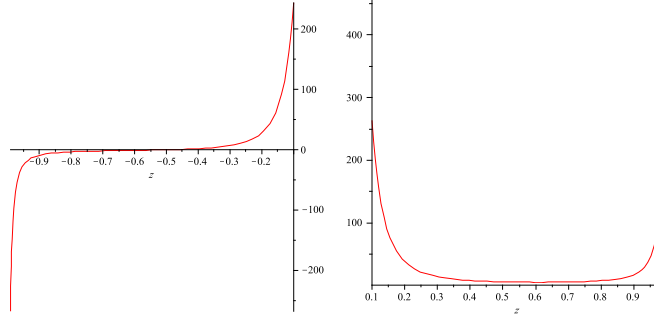


FIGURE 4. The graph of the function $f(z) = \frac{4z+|z|^{-1}}{4z^2(1-z^2)^{3/2}}$ in the intervals $(-1, 0)$ and $(0, 1)$, respectively.

Remark 5. If in Theorem 8 we take the masses $m_1 =: m$ and $m_2 = m_3 =: M$, the analogue of equation (52) is

$$\frac{4mz + M|z|^{-1}}{4z^2(1-z^2)^{3/2}} = \frac{\omega^2}{m}.$$

Then solutions exist for any $z \in (-\sqrt{M/m}/2, 0) \cup (0, 1)$. This means that there are no fixed points for $M \geq 4m$ (a fact that agrees with what we learned from Remark 2 and the proof of Theorem 1), so relative equilibria exist for such masses for all $z \in (-1, 0) \cup (0, 1)$.

6. RELATIVE EQUILIBRIA IN \mathbf{H}^2

In this section we will prove a few results about fixed points, as well as elliptic and hyperbolic relative equilibria in \mathbf{H}^2 . We also show that parabolic relative equilibria do not exist. Since, by the Principal Axis theorem for the Lorentz group, every Lorentzian rotation (see Appendix) can be written, in some basis, either as an elliptic rotation about the z axis, or as a hyperbolic rotation about the x axis, or as a parabolic rotation about the line $x = 0, y = z$, we can define three kinds of relative equilibria: the elliptic relative equilibria, the hyperbolic relative equilibria, and the parabolic relative equilibria. This terminology matches the standard terminology of hyperbolic geometry [35].

The elliptic relative equilibria are defined as follows.

Definition 3. *An elliptic relative equilibrium in \mathbf{H}^2 is a solution $\mathbf{q}_i = (x_i, y_i, z_i)$, $i = 1, \dots, n$, of equations (28) with $x_i = \rho_i \cos(\omega t + \alpha_i)$, $y_i = \rho_i \sin(\omega t + \alpha_i)$, and $z_i = (\rho_i^2 + 1)^{1/2}$, where ω, α_i , and ρ_i , $i = 1, \dots, n$, are constants.*

Remark that, as in \mathbf{S}^2 , a “weak” property of the center of mass occurs in \mathbf{H}^2 for elliptic relative equilibria. Indeed, if all the bodies are at all times on

one side of a plane containing the rotation axis, then the integrals of the angular momentum are violated because the vector representing the total angular momentum cannot be zero or parallel to the z axis.

Let us now define the hyperbolic relative equilibria.

Definition 4. A hyperbolic relative equilibrium in \mathbf{H}^2 is a solution of equations (28) of the form $\mathbf{q}_i = (x_i, y_i, z_i)$, $i = 1, \dots, n$, defined for all $t \in \mathbf{R}$, with

$$(53) \quad x_i = \text{constant}, \quad y_i = \rho_i \sinh(\omega t + \alpha_i), \quad \text{and} \quad z_i = \rho_i \cosh(\omega t + \alpha_i),$$

where ω, α_i , and $\rho_i = (1 + x_i^2)^{1/2} \geq 1$, $i = 1, \dots, n$, are constants.

Finally, the parabolic relative equilibria are defined as follows.

Definition 5. A parabolic relative equilibrium in \mathbf{H}^2 is a solution of equations (28) of the form $\mathbf{q}_i = (x_i, y_i, z_i)$, $i = 1, \dots, n$, defined for all $t \in \mathbf{R}$, with

$$(54) \quad \begin{aligned} x_i &= a_i - b_i t + c_i t \\ y_i &= a_i t + b_i(1 - t^2/2) + c_i t^2/2 \\ z_i &= a_i t - b_i t^2/2 + c_i(1 + t^2/2), \end{aligned}$$

where a_i, b_i and c_i , $i = 1, \dots, n$, are constants, and $a_i^2 + b_i^2 - c_i^2 = -1$.

6.1. Fixed Points in \mathbf{H}^2 . The simplest solutions of the equations of motion are the fixed points. They can be seen as trivial elliptic relative equilibria that correspond to $\omega = 0$. In terms of the equations of motion, we can define them as follows.

Definition 6. A solution of system (29) is called a fixed point if

$$\overline{\nabla}_{\mathbf{q}_i} U_{-1}(\mathbf{q})(t) = \mathbf{p}_i(t) = \mathbf{0} \quad \text{for all } t \in \mathbf{R} \quad \text{and} \quad i = 1, \dots, n.$$

Unlike in \mathbf{S}^2 , there are no fixed points in \mathbf{H}^2 . Let us formally state and prove this fact.

Theorem 9. In the n -body problem with $n \geq 2$ in \mathbf{H}^2 there are no configurations that correspond to fixed points of the equations of motion.

Proof. Consider any collisionless configuration of n bodies initially at rest in \mathbf{H}^2 . This means that the component of the forces acting on bodies due to the constraints, which involve the factors $\dot{x}_i^2 + \dot{y}_i^2 - \dot{z}_i^2$, $i = 1, \dots, n$, are zero at $t = 0$. At least one body, m_i , has the largest z coordinate. Notice that the interaction between m_i and any other body takes place along geodesics, which are concave-up hyperbolas on the ($z > 0$)-sheet of the hyperboloid modeling \mathbf{H}^2 . Then the body m_j , $j \neq i$, exercises an attraction on m_i down the geodesic hyperbola that connects these bodies, so the z coordinate of this force acting on m_i is negative, independently of whether $z_j(0) < z_i(0)$ or $z_j(0) = z_i(0)$. Since this is true for every $j = 1, \dots, n$, $j \neq i$, it follows that $\ddot{z}_i(0) < 0$.

Therefore m_i moves downwards the hyperboloid, so the original configuration is not a fixed point. \square

6.2. Elliptic Relative Equilibria in \mathbf{H}^2 . We now consider elliptic relative equilibria, and prove an analogue of Theorem 6.

Theorem 10. *Consider the n -body problem with equal masses in \mathbf{H}^2 . Then, for any $m > 0$ and $z > 1$, there are a positive and a negative ω that produce elliptic relative equilibria in which the bodies are at the vertices of an n -gon rotating in the plane $z = \text{constant}$.*

Proof. The proof works in the same way as for Theorem 6, by considering the cases n odd and even separately. The only differences are that we replace r with ρ , the relation $r^2 + z^2 = 1$ with $z^2 = \rho^2 + 1$, and the denominator $(1 - k_{0j}^2)^{3/2}$ with $(c_{0j}^2 - 1)^{3/2}$, wherever it appears, where $c_{0j} = -k_{0j}$ replaces k_{0j} . Unlike in \mathbf{S}^2 , the case n even is satisfied for all admissible values of z . \square

Like in \mathbf{S}^2 , the equilateral triangle presents particular interest, so let us say a bit more about it than in the general case of the regular n -gon.

Corollary 3. *Consider the 3-body with equal masses, $m := m_1 = m_2 = m_3$, in \mathbf{H}^2 . Then for any $m > 0$ and $z > 1$, there are a positive and a negative ω that produce relative elliptic equilibria in which the bodies are at the vertices of an equilateral triangle that rotates in the plane $z = \text{constant}$. Moreover, for every $\omega^2/m > 0$ there is a unique $z > 1$ as above.*

Proof. Substituting in system (28) a solution of the form

$$(55) \quad x_i = \rho \cos(\omega t + \alpha_i), \quad y_i = \rho \sin(\omega t + \alpha_i), \quad z_i = z,$$

with $z = \sqrt{\rho^2 + 1}$, $\alpha_1 = 0, \alpha_2 = 2\pi/3, \alpha_3 = 4\pi/3$, we are led to the equation

$$(56) \quad \frac{8}{\sqrt{3}(3z^4 - 2z^2 - 1)^{3/2}} = \frac{\omega^2}{m}.$$

The left hand side is positive for $z > 1$, tends to infinity when $z \rightarrow 1$, and tends to zero when $z \rightarrow \infty$. So for any z in this interval and $m > 0$, there are a positive and a negative ω for which the above equation is satisfied. \square

As we already proved in the previous section, an equilateral triangle rotating in its own plane forms an elliptic relative equilibrium in \mathbf{S}^2 only if the three masses lying at its vertices are equal. The same result is true in \mathbf{H}^2 , as we will further show.

Proposition 3. *In the 3-body problem in \mathbf{H}^2 , if the bodies m_1, m_2, m_3 are initially at the vertices of an equilateral triangle in the plane $z = \text{constant}$ for some $z > 1$, then there are initial velocities that lead to an elliptic relative equilibrium in which the triangle rotates in its own plane if and only if $m_1 = m_2 = m_3$.*

Proof. The implication which shows that if $m_1 = m_2 = m_3$, the rotating equilateral triangle is an elliptic relative equilibrium, follows from Theorem 10. To prove the other implication, we substitute into equations (28) a solution of the form (55) with $i = 1, 2, 3$, $\rho := \rho_1, \rho_2, \rho_3$, $z := z_1 = z_2 = z_3 = (\rho^2 + 1)^{1/2}$, and $\alpha_1 = 0, \alpha_2 = 2\pi/3, \alpha_3 = 4\pi/3$. The computations then lead to the system

$$(57) \quad \begin{cases} m_1 + m_2 = \zeta \omega^2 \\ m_2 + m_3 = \zeta \omega^2 \\ m_3 + m_1 = \zeta \omega^2, \end{cases}$$

where $\zeta = \sqrt{3}(3z^4 - 2z^2 - 1)^{3/2}/4$. But for any $z = \text{constant}$ with $z > 1$, the above system has a solution only for $m_1 = m_2 = m_3 = \zeta \omega^2/2$. Therefore the masses must be equal. \square

The following result perfectly resembles Theorem 7. The proof works the same way, by just replacing the elliptical trigonometric functions with hyperbolic ones and changing the signs to reflect the equations of motion in \mathbf{H}^2 .

Theorem 11. *For all Lagrangian solutions in \mathbf{H}^2 , the masses m_1, m_2 and m_3 have to rotate on the same circle, whose plane must be orthogonal to the rotation axis, and therefore $m_1 = m_2 = m_3$.*

We will further prove an analogue of Theorem 8.

Theorem 12. *Consider the 3-body problem in \mathbf{H}^2 with equal masses, $m := m_1 = m_2 = m_3$. Fix the body of mass m_1 at $(0, 0, 1)$ and the bodies of masses m_2 and m_3 at the opposite ends of a diameter on the circle $z = \text{constant}$. Then, for any $m > 0$ and $z > 1$, there are a positive and a negative ω , which produce elliptic relative equilibria that rotate around the z axis.*

Proof. Substituting into the equations of motion (28) a solution of the form

$$\begin{aligned} x_1 &= 0, & y_1 &= 0, & z_1 &= 1, \\ x_2 &= \rho \cos \omega t, & y_2 &= \rho \sin \omega t, & z_2 &= z, \\ x_3 &= \rho \cos(\omega t + \pi), & y_3 &= \rho \sin(\omega t + \pi), & z_3 &= z, \end{aligned}$$

where $\rho \geq 0$ and $z \geq 1$ are constants satisfying $z^2 = \rho^2 + 1$, leads either to identities or to the algebraic equation

$$(58) \quad \frac{4z^2 + 1}{4z^3(z^2 - 1)^{3/2}} = \frac{\omega^2}{m}.$$

The function on the left hand side is positive for $z > 1$. Therefore, for every $m > 0$ and $z > 1$, there are a positive and a negative ω that lead to a geodesic elliptic relative equilibrium. The sign of ω determines the sense of rotation. \square

Remark 6. For every $\omega^2/m > 0$, there is exactly one $z > 1$ that satisfies equation (58) (see Figure 5).

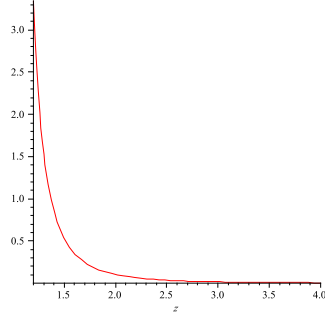


FIGURE 5. The graph of the function $f(z) = \frac{4z^2+1}{4z^3(z^2-1)^{3/2}}$ for $z > 1$.

6.3. Hyperbolic Relative Equilibria in \mathbf{H}^2 . We now prove some results concerning hyperbolic relative equilibria. We first show that, in the n -body problem, hyperbolic relative equilibria do not exist along any given fixed geodesic of \mathbf{H}^2 . In other words, the bodies cannot chase each other along a geodesic and maintain the same initial distances for all times.

Theorem 13. *Along any fixed geodesic, the n -body problem in \mathbf{H}^2 has no hyperbolic relative equilibria.*

Proof. Without loss of generality, we can prove this result for the geodesic $x = 0$. We will show that equations (28) do not have solutions of the form (53) with $x_i = 0$ and (consequently) $\rho_i = 1$, $i = 1, \dots, n$. Substituting

$$(59) \quad x_i = 0, \quad y_i = \sinh(\omega t + \alpha_i), \quad \text{and} \quad z_i = \cosh(\omega t + \alpha_i)$$

into system (28), the equation corresponding to the y_i coordinate becomes

$$(60) \quad \sum_{j=1, j \neq i}^n \frac{m_j [\sinh(\omega t + \alpha_j) - \cosh(\alpha_i - \alpha_j) \sinh(\omega t + \alpha_i)]}{|\sinh(\alpha_i - \alpha_j)|^3} = 0.$$

Assume now that $\alpha_i > \alpha_j$ for all $j \neq i$. Let $\alpha_{M(i)}$ be the maximum of all α_j with $j \neq i$. Then for $t \in (-\alpha_{M(i)}/\omega, -\alpha_i/\omega)$, we have that $\sinh(\omega t + \alpha_j) < 0$ for all $j \neq i$ and $\sinh(\omega t + \alpha_i) > 0$. Therefore the left hand side of equation (60) is negative in this interval, so the identity cannot take place for all $t \in \mathbf{R}$. It follows that a necessary condition to satisfy equation (60) is that $\alpha_{M(i)} \geq \alpha_i$. But this inequality must be verified for all $i = 1, \dots, n$, a fact that can be written as:

$$\begin{aligned} \alpha_1 &\geq \alpha_2 \quad \text{or} \quad \alpha_1 \geq \alpha_3 \quad \text{or} \quad \dots \quad \text{or} \quad \alpha_1 \geq \alpha_n, \\ \alpha_2 &\geq \alpha_1 \quad \text{or} \quad \alpha_2 \geq \alpha_3 \quad \text{or} \quad \dots \quad \text{or} \quad \alpha_2 \geq \alpha_n, \\ &\dots \\ \alpha_n &\geq \alpha_1 \quad \text{or} \quad \alpha_n \geq \alpha_2 \quad \text{or} \quad \dots \quad \text{or} \quad \alpha_n \geq \alpha_{n-1}. \end{aligned}$$

The constants $\alpha_1, \dots, \alpha_n$ must satisfy one inequality from each of the above lines. But every possible choice implies the existence of at least one i and one j with $i \neq j$ and $\alpha_i = \alpha_j$. For those i and j , $\sinh(\alpha_i - \alpha_j) = 0$, so equation (60) is undefined, therefore equations (28) cannot have solutions of the form (59). Consequently hyperbolic relative equilibria do not exist along the geodesic $x = 0$. \square

Theorem 13 raises the question whether hyperbolic relative equilibria do exist at all. For three equal masses, the answer is given by the following result, which shows that, in \mathbf{H}^2 , three bodies can move along hyperbolas lying in parallel planes of \mathbf{R}^3 , maintaining the initial distances among themselves and remaining on the same geodesic (which rotates hyperbolically). The existence of such solutions is surprising. They rather resemble fighter planes flying in formation than celestial bodies moving under the action of gravity alone.

Theorem 14. *In the 3-body problem of equal masses, $m := m_1 = m_2 = m_3$, in \mathbf{H}^2 , for any given $m > 0$ and $x \neq 0$, there exist a positive and a negative ω that lead to hyperbolic relative equilibria.*

Proof. We will show that $\mathbf{q}_i(t) = (x_i(t), y_i(t), z_i(t))$, $i = 1, 2, 3$, is a hyperbolic relative equilibrium of system (28) for

$$\begin{aligned} x_1 &= 0, & y_1 &= \sinh \omega t, & z_1 &= \cosh \omega t, \\ x_2 &= x, & y_2 &= \rho \sinh \omega t, & z_2 &= \rho \cosh \omega t, \\ x_3 &= -x, & y_3 &= \rho \sinh \omega t, & z_3 &= \rho \cosh \omega t, \end{aligned}$$

where $\rho = (1 + x^2)^{1/2}$. Notice first that

$$x_1 x_2 + y_1 y_2 - z_1 z_2 = x_1 x_3 + y_1 y_3 - z_1 z_3 = -\rho,$$

$$x_2 x_3 + y_2 y_3 - z_2 z_3 = -2x^2 - 1,$$

$$\dot{x}_1^2 + \dot{y}_1^2 - \dot{z}_1^2 = \omega^2, \quad \dot{x}_2^2 + \dot{y}_2^2 - \dot{z}_2^2 = \dot{x}_3^2 + \dot{y}_3^2 - \dot{z}_3^2 = \rho^2 \omega^2.$$

Substituting the above coordinates and expressions into equations (28), we are led either to identities or to the equation

$$(61) \quad \frac{4x^2 + 5}{4x^2|x|(x^2 + 1)^{3/2}} = \frac{\omega^2}{m},$$

from which the statement of the theorem follows. \square

Remark 7. The left hand side of equation (61) is undefined for $x = 0$, but it tends to infinity when $x \rightarrow 0$ and to 0 when $x \rightarrow \pm\infty$. This means that for each $\omega^2/m > 0$ there are exactly one positive and one negative x (equal in absolute value), which satisfy the equation.

Remark 8. Theorem 14 is also true if, say, $m := m_1$ and $M := m_2 = m_3$. Then the analogue of equation (61) is

$$\frac{m}{x^2|x|(x^2+1)^{1/2}} + \frac{M}{4x^2|x|(x^2+1)^{3/2}} = \omega^2,$$

and it is obvious that for any $m, M > 0$ and $x \neq 0$, there are a positive and negative ω satisfying the above equation.

Remark 9. Theorem 14 also works for two bodies of equal masses, $m := m_1 = m_2$, of coordinates

$$x_1 = -x_2 = x, y_1 = y_2 = \rho \sinh \omega t, z_1 = z_2 = \rho \cosh \omega t,$$

where x is a positive constant and $\rho = (x^2 + 1)^{3/2}$. Then the analogue of equation (61) is

$$\frac{1}{4x^2|x|(x^2+1)^{3/2}} = \frac{\omega^2}{m},$$

which obviously supports a statement similar to the one in Theorem 14.

6.4. Parabolic Relative Equilibria in \mathbf{H}^2 . We now show that there are no parabolic relative equilibria. More precisely, we prove the following result.

Theorem 15. *The n -body problem in \mathbf{H}^2 has no parabolic relative equilibria.*

Proof. Let x_i, y_i , and z_i be as in the definition of parabolic relative equilibria (54). Then $\dot{x}_i = -b_i + c_i$, $\dot{y}_i = a_i + (c_i - b_i)t$, and $\dot{z}_i = a_i + (c_i - b_i)t$. Thus, the first component of the angular momentum is $\sum_i m_i a_i (b_i - c_i) - \sum_i m_i (b_i - c_i)^2 t$. It follows that $b_i = c_i$ because the first component of the angular momentum must be constant. But $a_i^2 + b_i^2 - c_i^2 = -1$, hence $a_i^2 = -1$, which is impossible, since a_i is a real number. \square

7. SAARI'S CONJECTURE

In 1970, Don Saari conjectured that solutions of the classical n -body problem with constant moment of inertia are relative equilibria, [59], [60]. The moment of inertia is defined in classical Newtonian celestial mechanics as $\frac{1}{2} \sum_{i=1}^n m_i \mathbf{q}_i \cdot \mathbf{q}_i$, a function that gives a crude measure of the bodies' distribution in space. But this definition makes little sense in \mathbf{S}^2 and \mathbf{H}^2 because $\mathbf{q}_i \odot \mathbf{q}_i = \pm 1$ for every $i = 1, \dots, n$. To avoid this problem, we adopt the standard point of view used in physics, and define the moment of inertia in \mathbf{S}^2 or \mathbf{H}^2 about the direction of the angular momentum. But while fixing an axis in \mathbf{S}^2 does not restrain generality, the symmetry of \mathbf{H}^2 makes us distinguish between two cases.

Indeed, in \mathbf{S}^2 we can assume that the rotation takes place around the z axis, and thus define the moment of inertia as

$$(62) \quad \mathbf{I} := \sum_{i=1}^n m_i (x_i^2 + y_i^2).$$

In \mathbf{H}^2 , all possibilities can be reduced via suitable isometric transformations (see Appendix) to: (i) the symmetry about the z axis, when the moment of inertia takes the same form (62), and (ii) the symmetry about the x axis, which corresponds to hyperbolic rotations, when—in agreement with the definition of the Lorentz product (see Appendix)—we define the moment of inertia as

$$(63) \quad \mathbf{J} := \sum_{i=1}^n m_i (y_i^2 - z_i^2).$$

The case of the parabolic rotations will not be considered because there are no parabolic relative equilibria.

These definitions allow us to formulate the following conjecture:

Saari's Conjecture in \mathbf{S}^2 and \mathbf{H}^2 . *For the gravitational n -body problem in \mathbf{S}^2 and \mathbf{H}^2 , every solution that has a constant moment of inertia about the direction of the angular momentum is either an elliptic relative equilibrium in \mathbf{S}^2 or \mathbf{H}^2 , or a hyperbolic relative equilibrium in \mathbf{H}^2 .*

By generalizing an idea we used in the Euclidean case, [24], [25], we can now settle this conjecture when the bodies undergo another constraint. More precisely, we will prove the following result.

Theorem 16. *For the gravitational n -body problem in \mathbf{S}^2 and \mathbf{H}^2 , every solution with constant moment of inertia about the direction of the angular momentum for which the bodies remain aligned along a geodesic that rotates elliptically in \mathbf{S}^2 or \mathbf{H}^2 , or hyperbolically in \mathbf{H}^2 , is either an elliptic relative equilibrium in \mathbf{S}^2 or \mathbf{H}^2 , or a hyperbolic relative equilibrium in \mathbf{H}^2 .*

Proof. Let us first prove the case in which \mathbf{I} is constant in \mathbf{S}^2 and \mathbf{H}^2 , i.e. when the geodesic rotates elliptically. According to the above definition of \mathbf{I} , we can assume without loss of generality that the geodesic passes through the point $(0, 0, 1)$ and rotates about the z -axis with angular velocity $\omega(t) \neq 0$. The angular momentum of each body is $\mathbf{L}_i = m_i \mathbf{q}_i \otimes \dot{\mathbf{q}}_i$, so its derivative with respect to t takes the form

$$\dot{\mathbf{L}}_i = m_i \dot{\mathbf{q}}_i \otimes \dot{\mathbf{q}}_i + m_i \mathbf{q}_i \otimes \ddot{\mathbf{q}}_i = m_i \mathbf{q}_i \otimes \tilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q}) - m_i \dot{\mathbf{q}}_i^2 \mathbf{q}_i \otimes \mathbf{q}_i = m_i \mathbf{q}_i \otimes \tilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q}),$$

with $\kappa = 1$ in \mathbf{S}^2 and $\kappa = -1$ in \mathbf{H}^2 . Since $\mathbf{q}_i \odot \tilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = 0$, it follows that $\tilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q})$ is either zero or orthogonal to \mathbf{q}_i . (Recall that orthogonality here is meant in terms of the standard inner product because, both in \mathbf{S}^2 and \mathbf{H}^2 , $\mathbf{q}_i \odot \tilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = \mathbf{q}_i \cdot \nabla_{\mathbf{q}_i} U_\kappa(\mathbf{q})$.) If $\tilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = \mathbf{0}$, then $\dot{\mathbf{L}}_i = \mathbf{0}$, so $\dot{L}_i^z = 0$.

Assume now that $\tilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q})$ is orthogonal to \mathbf{q}_i . Since all the particles are on a geodesic, their corresponding position vectors are in the same plane, therefore any linear combination of them is in this plane, so $\tilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q})$ is in the same plane. Thus $\tilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q})$ and \mathbf{q}_i are in a plane orthogonal to the xy plane. It follows that $\dot{\mathbf{L}}_i$ is parallel to the xy plane and orthogonal to the z axis. Thus the z component, \dot{L}_i^z , of $\dot{\mathbf{L}}_i$ is 0, the same conclusion we obtained in the case $\tilde{\nabla}_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = \mathbf{0}$. Consequently, $L_i^z = c_i$, where c_i is a constant.

Let us also remark that since the angular momentum and angular velocity vectors are parallel to the z axis, $L_i^z = \mathbf{I}_i \omega(t)$, where $\mathbf{I}_i = m_i(x_i^2 + y_i^2)$ is the moment of inertia of the body m_i about the z -axis. Since the total moment of inertia, \mathbf{I} , is constant, and $\omega(t)$ is the same for all bodies because they belong to the same rotating geodesic, it follows that $\sum_{i=1}^n \mathbf{I}_i \omega(t) = \mathbf{I} \omega(t) = c$, where c is a constant. Consequently, ω is a constant vector.

Moreover, since $L_i^z = c_i$, it follows that $\mathbf{I}_i \omega(t) = c_i$. Then every \mathbf{I}_i is constant, and so is every z_i , $i = 1, \dots, n$. Hence each body of mass m_i has a constant z_i -coordinate, and all bodies rotate with the same constant angular velocity around the z -axis, properties that agree with our definition of an elliptic relative equilibrium.

We now prove the case $\mathbf{J} = \text{constant}$, i.e. when the geodesic rotates hyperbolically in \mathbf{H}^2 . According to the definition of \mathbf{J} , we can assume that the bodies are on a moving geodesic whose plane contains the x axis for all time and whose vertex slides along the geodesic hyperbola $x = 0$. (This moving geodesic hyperbola can be also visualized as the intersection between the sheet $z > 0$ of the hyperboloid and the plane containing the x axis and rotating about it. For an instant, this plane also contains the z axis.)

The angular momentum of each body is $\mathbf{L}_i = m_i \mathbf{q}_i \boxtimes \dot{\mathbf{q}}_i$, so we can show as before that its derivative takes the form $\dot{\mathbf{L}}_i = m_i \mathbf{q}_i \boxtimes \bar{\nabla}_{\mathbf{q}_i} U_{-1}(\mathbf{q})$. Again, $\bar{\nabla}_{\mathbf{q}_i} U_{-1}(\mathbf{q})$ is either zero or orthogonal to \mathbf{q}_i . In the former case we can draw the same conclusion as earlier, that $\dot{\mathbf{L}}_i = \mathbf{0}$, so in particular $\dot{L}_i^x = 0$. In the latter case, \mathbf{q}_i and $\bar{\nabla}_{\mathbf{q}_i} U_{-1}(\mathbf{q})$ are in the plane of the moving hyperbola, so their cross product, $\mathbf{q}_i \boxtimes \bar{\nabla}_{\mathbf{q}_i} U_{-1}(\mathbf{q})$ (which differs from the standard cross product only by its opposite z component), is orthogonal to the x axis, and therefore $\dot{L}_i^x = 0$. Thus $\dot{L}_i^x = 0$ in either case.

From here the proof proceeds as before by replacing \mathbf{I} with \mathbf{J} and the z axis with the x axis, and noticing that $L_i^x = \mathbf{J}_i \omega(t)$, to show that every m_i has a constant x_i coordinate. In other words, each body is moving along a (in general non-geodesic) hyperbola given by the intersection of the hyperboloid with a plane orthogonal to the x axis. These facts in combination with the sliding of the moving geodesic hyperbola along the fixed geodesic hyperbola $x = 0$ are in agreement with our definition of a hyperbolic relative equilibrium. \square

8. APPENDIX

8.1. The Weierstrass model. Since the Weierstrass model of the hyperbolic (Bolyai-Lobachevsky) plane is little known, we will present here its basic properties. This model appeals for at least two reasons: (i) it allows an obvious comparison with the sphere, both from the geometric and analytic point of view; (ii) it emphasizes the differences between the Bolyai-Lobachevsky and the Euclidean plane as clearly as the well-known differences between the Euclidean plane and the sphere. As far as we are concerned, this model was the key for obtaining the results we proved for the n -body problem for $\kappa < 0$.

The Weierstrass model is constructed on one of the sheets of the hyperboloid $x^2 + y^2 - z^2 = -1$ in the 3-dimensional Minkowski space $\mathcal{M}^3 := (\mathbf{R}^3, \square)$, in which $\mathbf{a} \square \mathbf{b} = a_x b_x + a_y b_y - a_z b_z$, with $\mathbf{a} = (a_x, a_y, a_z)$ and $\mathbf{b} = (b_x, b_y, b_z)$, represents the Lorentz inner product. We choose the $z > 0$ sheet of the hyperboloid, which we identify with the Bolyai-Lobachevsky plane \mathbf{H}^2 .

A linear transformation $T: \mathcal{M}^3 \rightarrow \mathcal{M}^3$ is orthogonal if $T(\mathbf{a}) \square T(\mathbf{a}) = \mathbf{a} \square \mathbf{a}$ for any $\mathbf{a} \in \mathcal{M}^3$. The set of these transformations, together with the Lorentz inner product, forms the orthogonal group $O(\mathcal{M}^3)$, given by matrices of determinant ± 1 . Therefore the group $SO(\mathcal{M}^3)$ of orthogonal transformations of determinant 1 is a subgroup of $O(\mathcal{M}^3)$. Another subgroup of $O(\mathcal{M}^3)$ is $G(\mathcal{M}^3)$, which is formed by the transformations T that leave \mathbf{H}^2 invariant. Furthermore, $G(\mathcal{M}^3)$ has the closed Lorentz subgroup, $\text{Lor}(\mathcal{M}^3) := G(\mathcal{M}^3) \cap SO(\mathcal{M}^3)$.

An important result is the Principal Axis Theorem for $\text{Lor}(\mathcal{M}^3)$, [26], [34]. Let us define the Lorentzian rotations about an axis as the 1-parameter subgroups of $\text{Lor}(\mathcal{M}^3)$ that leave the axis pointwise fixed. Then the Principal Axis Theorem states that every Lorentzian transformation has one of the forms:

$$A = P \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} P^{-1},$$

$$A = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh s & \sinh s \\ 0 & \sinh s & \cosh s \end{bmatrix} P^{-1},$$

or

$$A = P \begin{bmatrix} 1 & -t & t \\ t & 1 - t^2/2 & t^2/2 \\ t & -t^2/2 & 1 + t^2/2 \end{bmatrix} P^{-1},$$

where $\theta \in [0, 2\pi)$, $s, t \in \mathbf{R}$, and $P \in \text{Lor}(\mathcal{M}^3)$. These transformations are called elliptic, hyperbolic, and parabolic, respectively. The elliptic transformations are rotations about a *timelike* axis—the z axis in our case; hyperbolic rotations are rotations about a *spacelike* axis—the x axis in our context; and

parabolic transformations are rotations about a *lightlike* (or null) axis, represented here by the line $x = 0, y = z$. This result resembles Euler's Principal Axis Theorem, which states that any element of $SO(3)$ can be written, in some orthonormal basis, as a rotation about the z axis.

The geodesics of \mathbf{H}^2 are the hyperbolas obtained by intersecting the hyperboloid with planes passing through the origin of the coordinate system. For any two distinct points \mathbf{a} and \mathbf{b} of \mathbf{H}^2 , there is a unique geodesic that connects them, and the distance between these points is given by $d(\mathbf{a}, \mathbf{b}) = \cosh^{-1}(-\mathbf{a} \cdot \mathbf{b})$.

In the framework of Weierstrass's model, the parallels' postulate of hyperbolic geometry can be translated as follows. Take a geodesic γ , i.e. a hyperbola obtained by intersecting a plane through the origin, O , of the coordinate system with the upper sheet, $z > 0$, of the hyperboloid. This hyperbola has two asymptotes in its plane: the straight lines a and b , intersecting at O . Take a point, P , on the upper sheet of the hyperboloid but not on the chosen hyperbola. The plane aP produces the geodesic hyperbola α , whereas bP produces β . These two hyperbolas intersect at P . Then α and γ are parallel geodesics meeting at infinity along a , while β and γ are parallel geodesics meeting at infinity along b . All the hyperbolas between α and β (also obtained from planes through O) are non-secant with γ .

Like the Euclidean plane, the abstract Bolyai-Lobachevsky plane has no privileged points or geodesics. But the Weierstrass model has some convenient points and geodesics, such as the point $(0, 0, 1)$ and the geodesics passing through it. The elements of $\text{Lor}(\mathcal{M}^3)$ allow us to move the geodesics of \mathbf{H}^2 to convenient positions, a property we frequently use in this paper to simplify our arguments. Other properties of the Weierstrass model can be found in [29] and [58]. The Lorentz group is treated in some detail in [2], but the Principal Axis Theorems for the Lorentz group contained in [2] and [58] fails to include parabolic rotations, and is therefore incomplete.

8.2. History of the model. The first researcher who mentioned Karl Weierstrass in connection with the hyperboloidal model of the Bolyai-Lobachevsky plane was Wilhelm Killing. In a paper published in 1880, [40], he used what he called Weierstrass's coordinates to describe the "exterior hyperbolic plane" as an "ideal region" of the Bolyai-Lobachevsky plane. In 1885, he added that Weierstrass had introduced these coordinates, in combination with "numerous applications," during a seminar held in 1872, [42], pp. 258-259. We found no evidence of any written account of the hyperboloidal model for the Bolyai-Lobachevsky plane prior to the one Killing gave in a paragraph of [42], p. 260. His remarks might have inspired Richard Faber to name this model after Weierstrass and to dedicate a chapter to it in [29], pp. 247-278.

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